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Journal of Number Theory 123 (2007) 290–314
JOURNAL OF Number Theory
www.elsevier.com/locate/jnt

On approximation of real numbers by algebraic numbers of bounded degree

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Received 16 November 2001; revised 16 November 2005

Available online 7 September 2006

Communicated by A. Granville

Abstract

Dirichlet proved that for any real irrational number ξ there exist infinitely many rational numbers p/q such that $|\xi - p/q| < q^{-2}$. The correct generalization to the case of approximation by algebraic numbers of degree $\leqslant n$, $n > 2$, is still unknown. Here we prove a result which improves all previous estimates concerning this problem for $n > 2$.

1. Introduction

The problem of approximating real numbers by algebraic numbers is of classical interest in the theory of Diophantine approximation. In 1842 Dirichlet proved that for any real irrational number ξ there exist infinitely many rational numbers p/q such that

$$|\xi - p/q| < q^{-2}. \tag{1.1}$$

By the theorem of Hurwitz the upper bound q^{-2} can be replaced by $1/\sqrt{5}q^{-2}$ and in some sense this result is best possible. Denote by $\lceil P \rceil$ the height of a polynomial P, that is the largest absolute value of the coefficients of P. Multiplying (1.1) by q, we get $|q\xi - p| < q^{-1}$, and so we obtain the polynomial interpretation of Dirichlet's theorem, namely that for any real irrational number ξ there exist infinitely many polynomials P with integer coefficients of first degree such that $|P(\xi)| \ll \lceil P \rceil^{-1}$. Here $\ll$ is the Vinogradov symbol and the implicit constant depends on ξ only.

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0022-314X/$ – see front matter
doi:10.1016/j.jnt.2006.07.012

Let $\mathbb{A}_n$ denote the set of algebraic numbers of degree $\leqslant n$. Using Dirichlet's box principle, it is easy to prove a more general statement which claims that for any real number $\xi \notin \mathbb{A}_n$ there exist infinitely many polynomials P with integer coefficients of degree $\leqslant n$ such that $|P(\xi)| \ll \lceil P \rceil^{-n}$, where the implicit constant depends on ξ and n only. There are also complex and p-adic analogs of this theorem. For these reasons it is very natural to suppose that (1.1) can also be generalized to the case of approximation by algebraic numbers of degree $\leqslant n$. However, finding the correct generalization turns out to be very difficult.

Denote by $H(\alpha)$ the height of an algebraic number α, that is the largest absolute value of the coefficients of its minimal polynomial. In 1961 E. Wirsing [14] made the conjecture that for any real number $\xi \notin \mathbb{A}_n$ and any $\varepsilon > 0$ there exist infinitely many algebraic numbers $\alpha \in \mathbb{A}_n$ such that

$$|\xi - \alpha| \ll H(\alpha)^{-n-1+\varepsilon}, \tag{1.2}$$

where the implicit constant should depend on ξ, n, and ε only. Later W.M. Schmidt [7] conjectured that the optimal exponent in (1.2) is $-n-1$. These conjectures have not been resolved except in some special cases. V.G. Sprindžuk [8] showed that the conjecture of Wirsing is true for almost all real numbers. In [14] E. Wirsing also proved that for any real number $\xi \notin \mathbb{A}_n$ there exist infinitely many algebraic numbers $\alpha \in \mathbb{A}_n$ such that $|\xi - \alpha| \ll H(\alpha)^{-C(n)}$, where $\lim_{n\to\infty}(C(n) - n/2) = 2$ and the implicit constant depends on ξ and n only. He considered the complex case as well (the p-adic analog is contained in [6]). The basic idea consists in constructing infinitely many pairs of polynomials with integer coefficients of degree $\leqslant n$ that have "small" absolute values at ξ and have no common root. Considering their resultant gives us the desired result.

Another approach to this problem was described by H. Davenport and W.M. Schmidt [2]. They considered linearly independent linear forms $L(\mathbf{X})$ and $M(\mathbf{X})$ in three variables $\mathbf{X} = (x_1, x_2, x_3)$ and proved that there are infinitely many integer points $\mathbf{X}$ such that

$$\bigl|L(\mathbf{X})\bigr| \ll \bigl|M(\mathbf{X})\bigr| \lceil \mathbf{X} \rceil^{-3},$$

where $\lceil \mathbf{X} \rceil = \max(|x_1|, |x_2|, |x_3|)$ and the implicit constant depends on L and M only. To deduce the conjecture for $n = 2$, one has to put $L(\mathbf{X}) = x_1\xi^2 + x_2\xi + x_3$, $M(\mathbf{X}) = 2x_1\xi + x_2$ and use the well-known inequality

$$|\xi - \alpha| \leqslant n \frac{|G(\xi)|}{|G'(\xi)|}, \tag{1.3}$$

where G is a polynomial of degree n and α is the root of G closest to ξ. In [3,4] (see also [7]) H. Davenport and W.M. Schmidt obtained generalizations of their theorem. Unfortunately, these generalizations do not help to solve the problem for $n > 2$. Moreover, the investigations of these authors revealed a fundamental impossibility to proving the conjecture for $n > 2$ using arbitrary linear forms. In fact, in [3] they also showed that for any $k \geqslant 1$ and any $n \geqslant k + 2$ there exist linearly independent linear forms $L(\mathbf{X}), M_1(\mathbf{X}), \dots, M_k(\mathbf{X})$ in n variables $\mathbf{X} = (x_1, \dots, x_n)$ such that for every $\varepsilon > 0$ and every integer point $\mathbf{X}$ the following inequality holds:

$$\bigl|L(\mathbf{X})\bigr| \gg \max\bigl(\bigl|M_1(\mathbf{X})\bigr|, \dots, \bigl|M_k(\mathbf{X})\bigr|\bigr) \lceil \mathbf{X} \rceil^{-k-2-\varepsilon},$$

where $\boxed{\mathbf{X}} = \max(|x_1|, \dots, |x_n|)$ and the implicit constant depends on ε only. For a long period after this there were no new ideas or methods presented to help resolve the conjecture for $n > 2$ or even to improve the theorem of Wirsing.

In 1993 V.I. Bernik and K.I. Tsishchanka [1] obtained an improvement of Wirsing's result. They proved that for any real number $\xi \notin \mathbb{A}_n$ there exist infinitely many algebraic numbers $\alpha \in \mathbb{A}_n$ such that $|\xi - \alpha| \ll H(\alpha)^{-B(n)}$, where $\lim_{n\to\infty}(B(n) - n/2) = 3$ and the implicit constant depends on ξ and n only. The essence of the proof is to construct infinitely many polynomials Q with integer coefficients of degree $\leqslant n$ such that either $|Q(\xi)| \ll \boxed{Q}^{-\frac{n+\delta}{1+\delta}}$ and $|Q'(\xi)| \gg \boxed{Q}$ or $|Q(\xi)| \ll \boxed{Q}^{-n-\delta}$, where δ is some positive number. In the first case we use (1.3), whereas in the second case we apply Wirsing's method. Choosing the optimal value of δ, we get the desired result. The complex and p-adic analogs of this theorem are contained in [11] and [12], respectively.

In 1996 a new approach to this problem was introduced which became a useful starting point for further investigations. The first announcement of it was made in [9]. The paper [10] contains the most comprehensive description of the method. The idea can be interpreted in the following way. We first construct a sequence of n-tuples of linearly independent auxiliary polynomials with integer coefficients of degree $\leqslant n$ that have "small" absolute values at a given point. Then using them, we construct infinitely many polynomials with properties similar to Q from above.

In this paper, we develop the described method and prove a result which improves all previous estimates concerning the real case of this problem for $n > 2$.

Theorem. *Let n be an integer at least 3. Then for any real number $\xi \notin \mathbb{A}_n$ there exist infinitely many algebraic numbers $\alpha \in \mathbb{A}_n$ such that $|\xi - \alpha| \ll H(\alpha)^{-A(n)}$, where $A(n)$ is the largest real root of the polynomial*

$$T(x) = \begin{cases} 4x^5 - (4n+18)x^4 + (n^2+11n+30)x^3 \\ \quad - (2n^2+10n+22)x^2 + (2n^2+7n+4)x \\ \quad + n^2 - 5n + 2 & \text{if } n = 3, 4, 5, \\ 2x^5 - (n+12)x^4 + (2n+30)x^3 + (2n-41)x^2 \\ \quad - (3n-29)x + 2n - 10 & \text{if } n > 5. \end{cases} \tag{1.4}$$

The implicit constant depends on ξ and n only.

It can be shown (see Lemma 10.1) that

$$\lim_{n\to\infty}\bigl(A(n) - n/2\bigr) = 4.$$

Table 1 contains the approximate values of $C(n)$, $B(n)$, and $A(n)$ corresponding to Wirsing's theorem, the theorem from [1] and the theorem above, respectively.

From now on, n is a fixed integer at least 3. To shorten notation, we continue to write A instead of $A(n)$. The proof of the theorem will be indirect. Without loss of generality we can confine ourselves to the range $0 < \xi < 1/4$. So we assume that there exists a real number $\xi \notin \mathbb{A}_n$ in this range with the property that for any $c_1 > 0$ there is $H_1 > 0$ such that for all $\alpha \in \mathbb{A}_n$ with $H(\alpha) > H_1$ we have

$$|\xi - \alpha| > c_1 H(\alpha)^{-A}. \tag{1.5}$$

Table 1

n	$C(n)$	$B(n)$	$A(n)$
3	3.28	3.5	3.73
4	3.82	4.12	4.45
5	4.35	4.71	5.14
6	4.87	5.28	5.76
7	5.39	5.84	6.36
8	5.90	6.39	6.93
9	6.41	6.93	7.50
10	6.92	7.47	8.06
50	26.98	27.84	28.70
100	51.99	52.92	53.84

This will ultimately lead to a contradiction, and the theorem will follow. A detailed description of the idea of the proof will be given at the end of Section 4 after the key lemmas and constructions are introduced in Sections 2 and 3.

2. Auxiliary lemmas

In this section, we recall some relevant lemmas from [5,10,13].

Lemma 2.1. (See [10, Lemma 3.1].) *Let $G(x) = g_n x^n + \cdots + g_1 x + g_0$ be a polynomial with integer coefficients such that $|G(\xi)| < 1/2$. Then there is an index $j \in \{1, \ldots, n\}$ such that $|g_j| = \lceil G \rceil$.*

Lemma 2.2. (See [10, Lemma 3.2].) *Let G be a polynomial and j an index as in Lemma 2.1. Suppose $|g_i| \leqslant \xi^{n-1}\lceil G \rceil$ for every $i \in \{1, \ldots, n\} \setminus \{j\}$. Then $|G'(\xi)| > \xi^{n-1}\lceil G \rceil$.*

Lemma 2.3. (See [5, Lemma 2].) *Let $G, G_1, \ldots, G_k$ be polynomials such that $G = G_1 \cdots G_k$ and $\deg G = \ell$. Then*

$$e^{-\ell}\lceil G_1 \rceil \cdots \lceil G_k \rceil \leqslant \lceil G \rceil \leqslant (\ell+1)^{k-1}\lceil G_1 \rceil \cdots \lceil G_k \rceil. \tag{2.1}$$

Lemma 2.4. (See [10, Lemma 3.6].) *Let G_1, G_2 be polynomials with integer coefficients of degree $\leqslant \ell$. Let G_1 be irreducible over $\mathbb{Z}$ and $\lceil G_1 \rceil > e^{\ell}\lceil G_2 \rceil$. Then G_1 and G_2 have no common root.*

Lemma 2.5. *Let $G_1, G_2 \in \mathbb{Z}[x]$ be polynomials with $\deg G_1 = \ell$, $\deg G_2 = m$, $1 \leqslant \ell, m \leqslant n$. Suppose that G_1 and G_2 have no common root. If $\ell m \geqslant 2$, then at least one of the following estimates is true*:

$$\begin{aligned}
&\text{(i)} && 1 < c_2 \max\{|G_1(\xi)|, |G_2(\xi)|\}^2 \max\{\lceil G_1 \rceil, \lceil G_2 \rceil\}^{m+\ell-2},\\
&\text{(ii)} && 1 < c_2 \max\{|G_1(\xi)||G_1'(\xi)||G_2'(\xi)|, |G_2(\xi)||G_1'(\xi)|^2\}\lceil G_1 \rceil^{m-2}\lceil G_2 \rceil^{\ell-1},\\
&\text{(iii)} && 1 < c_2 \max\{|G_2(\xi)||G_1'(\xi)||G_2'(\xi)|, |G_1(\xi)||G_2'(\xi)|^2\}\lceil G_1 \rceil^{m-1}\lceil G_2 \rceil^{\ell-2},
\end{aligned} \tag{2.2}$$

where

$$c_2 = (2n)!\big((n+1)!\big)^{2n-2}.$$

If $\ell = m = 1$, *then*

$$1 \leqslant 2\max\{|G_1(\xi)|, |G_2(\xi)|\}\max\{\lceil G_1\rceil, \lceil G_2\rceil\}. \tag{2.3}$$

Proof. The first part of this lemma was already proved in [10] (see Lemma 3.4). To obtain (2.3), we put $G_1(x) = g_1^{(1)}x + g_0^{(1)}$, $G_2(x) = g_1^{(2)}x + g_0^{(2)}$. Then

$$1 \leqslant \left\| \begin{matrix} g_1^{(2)} & g_0^{(2)} \\ g_1^{(1)} & g_0^{(1)} \end{matrix} \right\| = \left\| \begin{matrix} g_1^{(2)} & G_2(\xi) \\ g_1^{(1)} & G_1(\xi) \end{matrix} \right\| \leqslant 2\max\{|G_1(\xi)|, |G_2(\xi)|\}\max\{\lceil G_1\rceil, \lceil G_2\rceil\}. \qquad \square$$

Lemma 2.6. (See [13, Theorem 1].) *Let*

$$\Lambda_\mu(\mathbf{X}) = \sum_{\nu=1}^{m} g_{\mu\nu}x_\nu, \quad \mu = 1, \ldots, k,$$

be k *linear forms in* m *variables* $\mathbf{X} = (x_1, \ldots, x_m)$. *Suppose that the forms* Λ_μ *are real for* $\mu = 1, \ldots, p$ *and that the remaining forms consist of* q *pairs of complex conjugate forms arranged so that* $\Lambda_{p+2t-1} = \overline{\Lambda}_{p+2t}$ *for* $t = 1, \ldots, q$. *Let also* S *be a positive integer and suppose that*

$$S^2\left\{\prod_{\nu=1}^{m}\alpha_\nu^{-2}\right\}\left\{\prod_{\mu=1}^{k}\left(1 + \beta_\mu^{-2}\sum_{\eta=1}^{m}\alpha_\eta^2|g_{\mu\eta}|^2\right)\right\} \leqslant 1,$$

where $\alpha_\nu \geqslant 1$ *for* $\nu = 1, \ldots, m$, $\beta_\mu > 0$ *for* $\mu = 1, \ldots, k$, *and* $\beta_{p+2t-1} = \beta_{p+2t}$ *for* $t = 1, \ldots, q$. *Then there exist* S *distinct pairs of nonzero lattice points*

$$\pm\mathbf{V}_s = \pm\begin{pmatrix} v_{1s} \\ \vdots \\ v_{ms} \end{pmatrix}, \quad s = 1, \ldots, S,$$

in $\mathbb{Z}^m$ *each of which satisfies the following conditions*:

$$|\Lambda_\mu(\pm\mathbf{V}_s)| \leqslant \beta_\mu, \quad \mu = 1, \ldots, p,$$
$$|\Lambda_\mu(\pm\mathbf{V}_s)| \leqslant \left(\frac{2}{\pi}\right)^{1/2}\beta_\mu, \quad \mu = p+1, \ldots, k,$$
$$|v_{\nu s}| \leqslant \alpha_\nu, \quad \nu = 1, \ldots, m.$$

3. Construction of auxiliary polynomials

In this section, the word "polynomial" shall mean a nonzero polynomial with integer coefficients of degree $\leqslant n$. Put

$$c_3 = \max(\xi^{-n+1}, n!).$$

We first construct a sequence of polynomials P_i such that

$$\text{(i)}\quad \frac{1}{2} > \left|P_1(\xi)\right| > c_3\left|P_2(\xi)\right| > \cdots > c_3^{i-1}\left|P_i(\xi)\right| > \cdots,$$
$$\text{(ii)}\quad \lceil P_1\rceil < \lceil P_2\rceil < \cdots < \lceil P_i\rceil < \cdots,$$
$$\text{(iii)}\quad \text{for any polynomial } P \text{ with } \lceil P\rceil < \lceil P_{i+1}\rceil \text{ we have } \left|P(\xi)\right| \geqslant c_3^{-1}\left|P_i(\xi)\right|. \tag{3.1}$$

Fix some $h \geqslant 1$. Consider the set of polynomials P with $\lceil P\rceil \leqslant h$. Their values at ξ are distinct. Hence there is a unique (up to the sign) polynomial P_1 with $\lceil P_1\rceil \leqslant h$ and minimal absolute value at ξ. Now increase h until a polynomial P_2 with $\lceil P_2\rceil \leqslant h$, $|P_2(\xi)| < c_3^{-1}|P_1(\xi)|$ appears. If there are several polynomials of this kind, pick one with minimal absolute value at ξ, etc. Repeating this process, we obtain the sequence of polynomials (3.1).

For any integer $i > 1$ we set $Q_i^{(0)}(x) = P_i(x)$. Write

$$Q_i^{(0)}(x) = b_n^{(0)}x^n + \cdots + b_1^{(0)}x + b_0^{(0)}.$$

By Lemma 2.1 there is an index $j_1 \in \{1, \ldots, n\}$ such that $|b_{j_1}^{(0)}| = \lceil Q_i^{(0)}\rceil$.

We now successively construct polynomials $Q_i^{(0)}, \ldots, Q_i^{(n-1)}$ and distinct integers $j_1, \ldots, j_n$ from $\{1, \ldots, n\}$. Write

$$Q_i^{(\ell)}(x) = b_n^{(\ell)}x^n + \cdots + b_1^{(\ell)}x + b_0^{(\ell)} \quad (\ell = 0, \ldots, n-1).$$

The polynomials $Q_i^{(\ell)}$ and the integers $j_{\ell+1}$ (which we call the *indices of the Q_i-system*) will have the following properties:

$$\text{(i)}\quad \left|Q_i^{(\ell)}(\xi)\right| < c_3^{-1}\left|P_{i-1}(\xi)\right|,$$
$$\text{(ii)}\quad \left|b_{j_\mu}^{(\ell)}\right| \leqslant c_3^{-1}\lceil Q_i^{(\mu-1)}\rceil \quad (\mu = 1, \ldots, \ell),$$
$$\text{(iii)}\quad \left|b_{j_{\ell+1}}^{(\ell)}\right| = \lceil Q_i^{(\ell)}\rceil,$$
$$\text{(iv)}\quad \lceil Q_i^{(\ell)}\rceil \leqslant c_3^{\frac{\ell+1}{n-\ell}}\left(\left|P_{i-1}(\xi)\right|\prod_{\nu=0}^{\ell-1}\lceil Q_i^{(\nu)}\rceil\right)^{-\frac{1}{n-\ell}} \tag{3.2}$$

(if $\ell = 0$, we have (3.2)(i) and (3.2)(iii) only). Moreover, if for some ℓ with $0 \leqslant \ell \leqslant n-1$ there is a polynomial $Q(x) = b_nx^n + \cdots + b_1x + b_0$ satisfying

$$\left|Q(\xi)\right| < c_3^{-1}\left|P_{i-1}(\xi)\right|,$$
$$|b_{j_\mu}| \leqslant c_3^{-1}\lceil Q_i^{(\mu-1)}\rceil \quad (\mu = 1, \ldots, \ell)$$

(if $\ell = 0$, we have $|Q(\xi)| < c_3^{-1}|P_{i-1}(\xi)|$ only), then $\lceil Q\rceil \geqslant \lceil Q_i^{(\ell)}\rceil$. In other words, $Q_i^{(\ell)}$ has minimum height among polynomials with (3.2)(i) and (3.2)(ii). We call this *the minimality property of $Q_i^{(\ell)}$*.

The pair $(Q_i^{(0)}, j_1)$ has the desired properties. Let t be some integer with $0 \leqslant t < n-1$ and suppose $(Q_i^{(0)}, j_1), \dots, (Q_i^{(t)}, j_{t+1})$ have been constructed so that (3.2)(i)–(iv) and the minimality property hold, and $j_1, \dots, j_{t+1}$ are distinct integers in $\{1, \dots, n\}$. We now construct $(Q_i^{(t+1)}, j_{t+2})$. By Minkowski's theorem on linear forms there is a polynomial $Q_i^{(t+1)}(x) = b_n^{(t+1)}x^n + \dots + b_1^{(t+1)}x + b_0^{(t+1)}$ having

$$\begin{aligned}
&\text{(i)} && \left|Q_i^{(t+1)}(\xi)\right| < c_3^{-1}\left|P_{i-1}(\xi)\right|, \\
&\text{(ii)} && \left|b_{j_\mu}^{(t+1)}\right| \leqslant c_3^{-1}\boxed{Q_i^{(\mu-1)}} \quad (\mu = 1, \dots, t+1), \\
&\text{(iii)} && \left|b_\eta^{(t+1)}\right| \leqslant c_3^{\frac{t+2}{n-t-1}}\left(\left|P_{i-1}(\xi)\right|\prod_{\nu=0}^{t}\boxed{Q_i^{(\nu)}}\right)^{-\frac{1}{n-t-1}}
\end{aligned} \tag{3.3}$$

for all $\eta \in \{1, \dots, n\} \setminus \{j_1, \dots, j_{t+1}\}$. If there are several polynomials of this kind, pick one whose height is minimal. By Lemma 2.1 there is an index $j_{t+2} \in \{1, \dots, n\}$ such that

$$\left|b_{j_{t+2}}^{(t+1)}\right| = \boxed{Q_i^{(t+1)}}. \tag{3.4}$$

We show that

$$j_{t+2} \in \{1, \dots, n\} \setminus \{j_1, \dots, j_{t+1}\}, \tag{3.5}$$

that is $j_1, \dots, j_{t+2}$ are distinct integers in $\{1, \dots, n\}$. In fact, by the minimality property we have $\boxed{Q_i^{(\mu-1)}} \leqslant \boxed{Q_i^{(t+1)}}$ for $\mu = 1, \dots, t+1$. Thanks to this, (3.3)(ii), (3.4) and the definition of c_3 we get

$$\left|b_{j_\mu}^{(t+1)}\right| \leqslant c_3^{-1}\boxed{Q_i^{(\mu-1)}} < \boxed{Q_i^{(\mu-1)}} \leqslant \boxed{Q_i^{(t+1)}} = \left|b_{j_{t+2}}^{(t+1)}\right|,$$

so $|b_{j_\mu}^{(t+1)}| < |b_{j_{t+2}}^{(t+1)}|$ for $\mu = 1, \dots, t+1$. This gives (3.5). Finally, (3.2)(iv) with $\ell = t+1$ follows from (3.3)(iii), (3.4) and (3.5). The arguments above imply that (3.2)(i)–(iv) with $\ell = t+1$ and the minimality property hold for $(Q_i^{(t+1)}, j_{t+2})$ and $j_1, \dots, j_{t+2}$ are distinct integers in $\{1, \dots, n\}$. In this way $(Q_i^{(0)}, j_1), \dots, (Q_i^{(n-1)}, j_n)$ are constructed.

By (3.1)(ii) and the minimality property we have

$$\boxed{P_{i-1}} < \boxed{Q_i^{(0)}} \leqslant \boxed{Q_i^{(1)}} \leqslant \dots \leqslant \boxed{Q_i^{(n-1)}}. \tag{3.6}$$

Lemma 3.1. *For any integer $i > 1$ the polynomials $P_{i-1}, Q_i^{(0)}, \dots, Q_i^{(n-2)}$ are linearly independent.*

Proof. Put

$$D = \begin{vmatrix} P_{i-1}(\xi) & Q_i^{(0)}(\xi) & \dots & Q_i^{(n-2)}(\xi) \\ a_{j_1} & b_{j_1}^{(0)} & \dots & b_{j_1}^{(n-2)} \\ \vdots & \vdots & & \vdots \\ a_{j_{n-1}} & b_{j_{n-1}}^{(0)} & \dots & b_{j_{n-1}}^{(n-2)} \end{vmatrix}, \qquad d = \begin{vmatrix} b_{j_1}^{(0)} & \dots & b_{j_1}^{(n-2)} \\ \vdots & & \vdots \\ b_{j_{n-1}}^{(0)} & \dots & b_{j_{n-1}}^{(n-2)} \end{vmatrix},$$

where $a_{j_1}, \dots, a_{j_{n-1}}$ are the coefficients of P_{i-1}. Let μ and ℓ be some integers with $1 \leqslant \mu \leqslant \ell \leqslant n-2$. By (3.2)(ii), (3.6) and the definition of c_3 we have

$$\left|b_{j_\mu}^{(\ell)}\right| \leqslant c_3^{-1} \left\lceil Q_i^{(\mu-1)} \right\rceil \leqslant c_3^{-1} \left\lceil Q_i^{(\ell)} \right\rceil \leqslant \frac{1}{n!} \left\lceil Q_i^{(\ell)} \right\rceil,$$

hence

$$|d| > \prod_{\nu=0}^{n-2} \left|b_{j_{\nu+1}}^{(\nu)}\right| - \frac{1}{n} \prod_{\nu=0}^{n-2} \left\lceil Q_i^{(\nu)} \right\rceil = \frac{n-1}{n} \prod_{\nu=0}^{n-2} \left\lceil Q_i^{(\nu)} \right\rceil \tag{3.7}$$

by (3.2)(iii). On the other hand, by (3.6) the absolute values of other minors from the last $n-1$ rows of D are $< (n-1)! \prod_{\nu=0}^{n-2} \left\lceil Q_i^{(\nu)} \right\rceil$. By this, (3.2)(i), (3.7) and the definition of c_3 we get

$$\begin{aligned} |D| &> \frac{n-1}{n} \left|P_{i-1}(\xi)\right| \prod_{\nu=0}^{n-2} \left\lceil Q_i^{(\nu)} \right\rceil - (n-1)! \left(\sum_{\nu=0}^{n-2} \left|Q_i^{(\nu)}(\xi)\right| \right) \prod_{\nu=0}^{n-2} \left\lceil Q_i^{(\nu)} \right\rceil \\ &> \frac{n-1}{n} \left|P_{i-1}(\xi)\right| \prod_{\nu=0}^{n-2} \left\lceil Q_i^{(\nu)} \right\rceil - \frac{n-1}{n} \left|P_{i-1}(\xi)\right| \prod_{\nu=0}^{n-2} \left\lceil Q_i^{(\nu)} \right\rceil \\ &= 0. \end{aligned}$$

So, $D \neq 0$, therefore the polynomials $P_{i-1}, Q_i^{(0)}, \dots, Q_i^{(n-2)}$ are linearly independent. □

4. Construction of polynomials $L_{i,\tau}$

Let i and τ be integers greater than 1 and let $\kappa_1, \dots, \kappa_{n-1}$ be the indices of the Q_τ-system. Consider the following system of n inequalities:

$$\begin{aligned} &\left| P_{i-1}(\xi) x_1 + \sum_{\nu=0}^{n-2} Q_i^{(\nu)}(\xi) x_{\nu+2} \right| \leqslant (2n)^{n/2} c_3^{n-1} \left|P_{i-1}(\xi)\right| \prod_{\nu=0}^{n-2} \left\lceil Q_i^{(\nu)} \right\rceil \prod_{\nu=0}^{n-2} \left\lceil Q_\tau^{(\nu)} \right\rceil^{-1}, \\ &\left| a_{\kappa_1} x_1 + \sum_{\nu=0}^{n-2} b_{\kappa_1}^{(\nu)} x_{\nu+2} \right| \leqslant c_3^{-1} \left\lceil Q_\tau^{(0)} \right\rceil, \\ &\quad \vdots \\ &\left| a_{\kappa_{n-1}} x_1 + \sum_{\nu=0}^{n-2} b_{\kappa_{n-1}}^{(\nu)} x_{\nu+2} \right| \leqslant c_3^{-1} \left\lceil Q_\tau^{(n-2)} \right\rceil. \end{aligned} \tag{4.1}$$

Lemma 4.1. *The system* (4.1) *has a nonzero integer solution* $(\tilde{x}_1, \dots, \tilde{x}_n)$ *which satisfies the following conditions*:

$$|\tilde{x}_1| \leqslant R\lceil P_{i-1}\rceil^{-1}, \qquad |\tilde{x}_\nu| \leqslant R\lceil Q_i^{(\nu-2)}\rceil^{-1} \quad (\nu = 2, \dots, n), \tag{4.2}$$

where

$$R = \max\left\{2^{n/2} n^{(n-1)/2} c_3^{n-1} \prod_{\nu=0}^{n-2} \lceil Q_i^{(\nu)}\rceil \prod_{\nu=0}^{n-2} \lceil Q_\tau^{(\nu)}\rceil^{-1} \lceil P_{i-1}\rceil,\ n^{-1/2} c_3^{-1} \lceil Q_\tau^{(n-2)}\rceil\right\}. \tag{4.3}$$

Proof. We apply Lemma 2.6 with $S = 1$, $k = m = p = n$, $q = 0$ and

$$\begin{aligned}
&\alpha_1 = R\lceil P_{i-1}\rceil^{-1}, \qquad \alpha_\nu = R\lceil Q_i^{(\nu-2)}\rceil^{-1} \quad (\nu = 2, \dots, n),\\
&\beta_1 = (2n)^{n/2} c_3^{n-1} |P_{i-1}(\xi)| \prod_{\nu=0}^{n-2} \lceil Q_i^{(\nu)}\rceil \prod_{\nu=0}^{n-2} \lceil Q_\tau^{(\nu)}\rceil^{-1},\\
&\beta_\mu = c_3^{-1} \lceil Q_\tau^{(\mu-2)}\rceil \quad (\mu = 2, \dots, n),\\
&g_{11} = P_{i-1}(\xi), \qquad g_{1\nu} = Q_i^{(\nu-2)}(\xi) \quad (\nu = 2, \dots, n),\\
&g_{\mu 1} = a_{\kappa_{\mu-1}} \quad (\mu = 2, \dots, n),\\
&g_{\mu\nu} = b_{\kappa_{\mu-1}}^{(\nu-2)} \quad (\nu = 2, \dots, n,\ \mu = 2, \dots, n).
\end{aligned} \tag{4.4}$$

We claim that

$$\left\{\prod_{\nu=1}^{n} \alpha_\nu^{-2}\right\}\left\{\prod_{\mu=1}^{n}\left(1 + \beta_\mu^{-2} \sum_{\eta=1}^{n} \alpha_\eta^2 |g_{\mu\eta}|^2\right)\right\} \leqslant 1. \tag{4.5}$$

In fact, by (4.4) we get

$$\prod_{\nu=1}^{n} \alpha_\nu^{-2} = R^{-2n} \lceil P_{i-1}\rceil^2 \prod_{\nu=0}^{n-2} \lceil Q_i^{(\nu)}\rceil^2. \tag{4.6}$$

To estimate the second term of (4.5), we note that by (3.2)(i), (3.6) and (4.4) we have

$$\alpha_\eta \leqslant \alpha_1, \quad |g_{1\eta}| \leqslant |g_{11}| \quad \text{and} \quad \alpha_\eta |g_{\mu\eta}| \leqslant R,$$

where $\eta = 1, \dots, n$ and $\mu = 2, \dots, n$. Therefore

$$\begin{aligned}
\prod_{\mu=1}^{n}\left(1 + \beta_\mu^{-2} \sum_{\eta=1}^{n} \alpha_\eta^2 |g_{\mu\eta}|^2\right) &= \left(1 + \beta_1^{-2} \sum_{\eta=1}^{n} \alpha_\eta^2 |g_{1\eta}|^2\right) \prod_{\mu=2}^{n}\left(1 + \beta_\mu^{-2} \sum_{\eta=1}^{n} \alpha_\eta^2 |g_{\mu\eta}|^2\right)\\
&\leqslant \left(1 + n\beta_1^{-2} \alpha_1^2 |g_{11}|^2\right) \prod_{\mu=2}^{n} \left(1 + n\beta_\mu^{-2} R^2\right).
\end{aligned} \tag{4.7}$$

We also note that

$$R = \max\left\{n^{-1/2}\beta_1\overline{|P_{i-1}|}\left|P_{i-1}(\xi)\right|^{-1},\ n^{-1/2}\beta_n\right\}$$

by (4.3) and (4.4). From this, (3.6) and (4.4) it follows that

$$n\beta_1^{-2}\alpha_1^2|g_{11}|^2 = n\beta_1^{-2}R^2\overline{|P_{i-1}|}^{-2}\left|P_{i-1}(\xi)\right|^2 \geqslant 1$$

and

$$n\beta_\mu^{-2}R^2 \geqslant n\beta_n^{-2}R^2 \geqslant 1 \quad (\mu = 2, \ldots, n).$$

Applying these inequalities to (4.7) and then using (4.4), we obtain

$$\prod_{\mu=1}^{n}\left(1+\beta_\mu^{-2}\sum_{\eta=1}^{n}\alpha_\eta^2|g_{\mu\eta}|^2\right) \leqslant 2n\beta_1^{-2}\alpha_1^2|g_{11}|^2\prod_{\mu=2}^{n}\left(2n\beta_\mu^{-2}R^2\right) = R^{2n}\overline{|P_{i-1}|}^{-2}\prod_{\nu=0}^{n-2}\overline{\left|Q_i^{(\nu)}\right|}^{-2}.$$

This and (4.6) give (4.5). □

We now are in the right position to outline the proof of the theorem. Consider the following polynomial:

$$L_{i,\tau}(x) = P_{i-1}(x)\tilde{x}_1 + \sum_{\nu=0}^{n-2} Q_i^{(\nu)}(x)\tilde{x}_{\nu+2}, \tag{4.8}$$

where, as before, $(\tilde{x}_1, \ldots, \tilde{x}_n)$ is a nonzero integer solution of the system (4.1) which satisfies (4.2). From this and Lemma 3.1 it follows that $L_{i,\tau}$ is nonzero and has integer coefficients. The main goal of the next four sections is to prove that there is an integer k_0 such that if $i > \tau \geqslant k_0$ and

$$\overline{|P_{i-1}|} \leqslant c_4\overline{|P_\tau|}, \tag{4.9}$$

where

$$c_4 = \left(2(2n)^{n/2}c_3^n\right)^{\frac{A-2}{n-1}},$$

then

$$\left|L_{i,\tau}(\xi)\right| < \left|L'_{i,\tau}(\xi)\right|^{-A+1}. \tag{4.10}$$

To this end we first deduce the upper bounds for $|P_{i-1}(\xi)|$, $\prod_{\nu=0}^{n-2}\overline{\left|Q_i^{(\nu)}\right|}$ and $\overline{\left|Q_i^{(n-2)}\right|}$ (see Section 5). Using these estimates and Lemma 4.1, we obtain the upper bounds for $|L_{i,\tau}(\xi)|$ and $|L'_{i,\tau}(\xi)|$ in Section 6. To prove (4.10), we also need the lower bounds for $|P_{i-1}(\xi)|$ which we derive in Section 7. In Section 8, we combine all these estimates and deduce (4.10). In Section 9, we use (4.10) to prove that

$$\left|P_{\tau-1}(\xi)\right| \prod_{\nu=0}^{n-2} \boxed{Q_\tau^{(\nu)}} \leqslant (2n)^{n/2} c_3^n \left|P_{i-1}(\xi)\right| \prod_{\nu=0}^{n-2} \boxed{Q_i^{(\nu)}}$$

and then we show that this leads us to a contradiction. This will complete the proof of the theorem. Finally, the last section contains the necessary explanations concerning the calculations from this paper.

5. Upper bounds for $|P_{i-1}(\xi)|$, $\prod_{\nu=0}^{n-2} \boxed{Q_i^{(\nu)}}$ and $\boxed{Q_i^{(n-2)}}$

Before we deduce the estimates, several observations are necessary. It follows from (1.3) and (1.5) that there exists c_5 with $0 < c_5 < 1$ such that

$$\left|G'(\xi)\right| < c_5^{-1}\left|G(\xi)\right|\boxed{G}^A \tag{5.1}$$

for any $G \in \mathbb{Z}[x]$, $G \not\equiv 0$, $\deg G \leqslant n$. Put

$$\omega = \omega(n) = (n-1)\frac{A-1}{A-2}.$$

Note that ω is well defined by (10.1)(iii). Put also

$$c_6 = c_2^{1/2}\left(\xi^{n-1} c_3^{1-n(A-1)} c_5\right)^{-\frac{1}{A-2}} e^{n\omega},$$
$$c_7 = \min\left\{\left|P(\xi)\right|:\ P \in \mathbb{Z}[x],\ P \not\equiv 0,\ \deg P \leqslant n,\ \boxed{P} \leqslant \max\left\{c_6, e^n \boxed{P_1}\right\}\right\}$$

and

$$c_8 = \max\Big\{\xi^{-n+1}(2n)^{n(A-1)/2} c_3^{(2n-1)(A-1)-1} c_4^{(n-1)(A-1)},\ c_6^2 \max\{c_5^{-2} c_6^2,\ c_7^{-1}\}^{n-1} e^{n(2n-1)}\Big\}. \tag{5.2}$$

It follows from (1.3) and (1.5) that there exists $H_2 > 0$ such that

$$\left|G'(\xi)\right| < c_8^{-1}\left|G(\xi)\right|\boxed{G}^A \tag{5.3}$$

for any $G \in \mathbb{Z}[x]$, $\deg G \leqslant n$, $\boxed{G} > H_2$. From now on, H_2 is a fixed number. By (5.1) and (5.3) we have

$$\left|G'(\xi)\right| < \delta^{-1}\left|G(\xi)\right|\boxed{G}^A \tag{5.4}$$

for any $G \in \mathbb{Z}[x]$, $G \not\equiv 0$, $\deg G \leqslant n$, where

$$\delta = \delta(G) = \begin{cases} c_5 & \text{if } \boxed{G} \leqslant H_2, \\ c_8 & \text{if } \boxed{G} > H_2. \end{cases}$$

Lemma 5.1. *Let i be an integer > 1. Then*

$$\left|P_{i-1}(\xi)\right| < \left(\xi^{n-1} c_3^{1-n(A-1)} \delta\right)^{-\frac{1}{A-2}} \overline{|P_i|}^{-\omega}, \tag{5.5}$$

where $\delta = \delta(P_i)$. Suppose i_0 is an integer > 1 such that $\overline{|P_{i_0}|} > H_2$. Then for any $i \geqslant i_0$ we have

$$\begin{aligned}
&\text{(i)} && \left|P_{i-1}(\xi)\right| < \overline{|P_i|}^{-\omega}, \\
&\text{(ii)} && \left|P_{i-1}(\xi)\right| < \overline{|P_i|}^{-n}, \\
&\text{(iii)} && \prod_{\nu=0}^{n-2} \overline{\left|Q_i^{(\nu)}\right|} < (2n)^{-n/2} (c_3 c_4)^{-n+1} \left|P_{i-1}(\xi)\right|^{-\frac{A-2}{A-1}}, \\
&\text{(iv)} && \overline{\left|Q_i^{(n-2)}\right|} < \left|P_{i-1}(\xi)\right|^{-\frac{A-2}{A-1}} \overline{|P_i|}^{-n+2}.
\end{aligned} \tag{5.6}$$

Suppose i and τ are integers such that $i > \tau \geqslant i_0$ and (4.9) holds. Then

$$\overline{\left|Q_\tau^{(n-2)}\right|} < \left|P_{i-1}(\xi)\right|^{-\frac{A-2}{A-1}} \overline{|P_{i-1}|}^{-n+2}. \tag{5.7}$$

Proof. We first note that $Q_i^{(n-1)}$ satisfies the conditions of Lemma 2.2 by (3.2)(ii) with $\ell = n-1$, (3.6) and the definition of c_3. From this lemma, (3.2)(i) and (5.4) it follows that

$$\xi^{n-1} \overline{\left|Q_i^{(n-1)}\right|} < \left|Q_i^{(n-1)'}(\xi)\right| < \delta^{-1} \left|Q_i^{(n-1)}(\xi)\right| \overline{\left|Q_i^{(n-1)}\right|}^{A} < c_3^{-1} \delta^{-1} \left|P_{i-1}(\xi)\right| \overline{\left|Q_i^{(n-1)}\right|}^{A},$$

where $\delta = \delta(Q_i^{(n-1)})$. Hence

$$\overline{\left|Q_i^{(n-1)}\right|}^{-A+1} < \xi^{-n+1} c_3^{-1} \delta^{-1} \left|P_{i-1}(\xi)\right|.$$

By this and (3.2)(iv) with $\ell = n-1$ we have

$$c_3^{-n(A-1)} \left(\left|P_{i-1}(\xi)\right| \prod_{\nu=0}^{n-2} \overline{\left|Q_i^{(\nu)}\right|}\right)^{A-1} < \xi^{-n+1} c_3^{-1} \delta^{-1} \left|P_{i-1}(\xi)\right|. \tag{5.8}$$

Using (3.6) in (5.8), we obtain

$$c_3^{-n(A-1)} \left(\left|P_{i-1}(\xi)\right| \overline{|P_i|}^{n-1}\right)^{A-1} < \xi^{-n+1} c_3^{-1} \delta^{-1} \left|P_{i-1}(\xi)\right|. \tag{5.9}$$

Since $\overline{\left|Q_i^{(n-1)}\right|} \geqslant \overline{|P_i|}$ by (3.6), it follows that (5.9) can be rewritten as (5.5) by the definition of δ.

To deduce (5.6)(i) from (5.5), we note that $\delta = c_8$, since $\overline{|P_i|} > H_2$ by the assumption above. This gives the desired result by (5.2). The estimate (5.6)(ii) immediately follows from (5.6)(i), since $\omega > n$ by the definition of ω and (10.1)(iii).

To obtain (5.6)(iii), we rewrite (5.8) as

$$\prod_{\nu=0}^{n-2} \overline{\left|Q_i^{(\nu)}\right|} < \left(\xi^{n-1} c_3^{1-n(A-1)} \delta\right)^{-\frac{1}{A-1}} \left|P_{i-1}(\xi)\right|^{-\frac{A-2}{A-1}},$$

and the result follows by (5.2), since $\delta = c_8$. If we rewrite (5.6)(iii) as

$$\overline{\left|Q_i^{(n-2)}\right|} < (2n)^{-n/2} (c_3 c_4)^{-n+1} \left|P_{i-1}(\xi)\right|^{-\frac{A-2}{A-1}} \prod_{\nu=0}^{n-3} \overline{\left|Q_i^{(\nu)}\right|}^{-1}$$

and apply (3.6) to the right-hand side, we get (5.6)(iv) by the definitions of c_3 and c_4. Similarly, since $\tau \geqslant i_0$, we have

$$\overline{\left|Q_\tau^{(n-2)}\right|} < (2n)^{-n/2} (c_3 c_4)^{-n+1} \left|P_{\tau-1}(\xi)\right|^{-\frac{A-2}{A-1}} \prod_{\nu=0}^{n-3} \overline{\left|Q_\tau^{(\nu)}\right|}^{-1},$$

which is

$$\leqslant (2n)^{-n/2} (c_3 c_4)^{-n+1} \left|P_{\tau-1}(\xi)\right|^{-\frac{A-2}{A-1}} \overline{\left|P_\tau\right|}^{-n+2}$$

by (3.6). Since $i > \tau$, this gives (5.7) by (3.1)(i), (4.9) and the definitions of c_3, c_4. □

6. Upper bounds for $|L_{i,\tau}(\xi)|$ and $|L'_{i,\tau}(\xi)|$

Lemma 6.1. *Let i_0 be an integer as in Lemma* 5.1. *Suppose i and τ are integers such that $i > \tau \geqslant i_0$ and* (4.9) *holds. Then the following estimates are valid*:

$$\begin{aligned} &\text{(i)} \quad \left|L_{i,\tau}(\xi)\right| < \left|P_{i-1}(\xi)\right|^{\frac{1}{A-1}} \overline{\left|P_{i-1}\right|}^{-n+1}, \\ &\text{(ii)} \quad \left|L'_{i,\tau}(\xi)\right| < \left|P_{i-1}(\xi)\right|^{3-A-\frac{A-2}{A-1}} \overline{\left|P_i\right|}^{-(n-2)(A-1)} \overline{\left|P_{i-1}\right|}^{-n+2}. \end{aligned} \tag{6.1}$$

Proof. By (4.1) and (4.8) we have

$$\left|L_{i,\tau}(\xi)\right| \leqslant (2n)^{n/2} c_3^{n-1} \left|P_{i-1}(\xi)\right| \prod_{\nu=0}^{n-2} \overline{\left|Q_i^{(\nu)}\right|} \prod_{\nu=0}^{n-2} \overline{\left|Q_\tau^{(\nu)}\right|}^{-1}.$$

Applying (5.6)(iii) to $\prod_{\nu=0}^{n-2} \overline{\left|Q_i^{(\nu)}\right|}$ and (3.6) with (4.9) to $\prod_{\nu=0}^{n-2} \overline{\left|Q_\tau^{(\nu)}\right|}^{-1}$, we obtain (6.1)(i).

We now estimate $|L'_{i,\tau}(\xi)|$. Using (3.2)(i), (3.6), (4.2), (4.8), (5.3) and the definitions of c_3, c_8, we get

$$\begin{aligned} \left|L'_{i,\tau}(\xi)\right| &\leqslant \left|P'_{i-1}(\xi)\right| |\tilde{x}_1| + \sum_{\nu=0}^{n-2} \left|Q_i^{(\nu)'}(\xi)\right| |\tilde{x}_{\nu+2}| \\ &< c_8^{-1} R \left(\left|P_{i-1}(\xi)\right| \overline{\left|P_{i-1}\right|}^{A-1} + \sum_{\nu=0}^{n-2} \left|Q_i^{(\nu)}(\xi)\right| \overline{\left|Q_i^{(\nu)}\right|}^{A-1} \right) \end{aligned}$$

$$< c_8^{-1} R \left|P_{i-1}(\xi)\right| \left(\lceil P_{i-1} \rceil^{A-1} + c_3^{-1} \sum_{\nu=0}^{n-2} \lceil Q_i^{(\nu)} \rceil^{A-1} \right)$$

$$< R \left|P_{i-1}(\xi)\right| \lceil Q_i^{(n-2)} \rceil^{A-1}.$$

This and (5.6)(iv) imply

$$\left|L'_{i,\tau}(\xi)\right| < R \left|P_{i-1}(\xi)\right|^{3-A} \lceil P_i \rceil^{-(n-2)(A-1)}. \tag{6.2}$$

We now show that

$$R < \left|P_{i-1}(\xi)\right|^{-\frac{A-2}{A-1}} \lceil P_{i-1} \rceil^{-n+2}. \tag{6.3}$$

We first note that

$$R < \max\left\{ (2n)^{n/2} c_3^{n-1} \prod_{\nu=0}^{n-2} \lceil Q_i^{(\nu)} \rceil \prod_{\nu=0}^{n-2} \lceil Q_\tau^{(\nu)} \rceil^{-1} \lceil P_{i-1} \rceil, \quad \lceil Q_\tau^{(n-2)} \rceil \right\} \tag{6.4}$$

by (4.3) and the definition of c_3. We have

$$(2n)^{n/2} c_3^{n-1} \prod_{\nu=0}^{n-2} \lceil Q_i^{(\nu)} \rceil \prod_{\nu=0}^{n-2} \lceil Q_\tau^{(\nu)} \rceil^{-1} \lceil P_{i-1} \rceil \leqslant (2n)^{n/2} (c_3 c_4)^{n-1} \prod_{\nu=0}^{n-2} \lceil Q_i^{(\nu)} \rceil \lceil P_{i-1} \rceil^{-n+2}$$

$$< \left|P_{i-1}(\xi)\right|^{-\frac{A-2}{A-1}} \lceil P_{i-1} \rceil^{-n+2} \tag{6.5}$$

by (3.6), (4.9) and (5.6)(iii). Clearly, (6.3) follows from (5.7), (6.4) and (6.5). Combining (6.2) with (6.3), we get (6.1)(ii). □

7. Lower bounds for $|P_{i-1}(\xi)|$

Put

$$\Phi(x) = \Phi(x, n) = \max\left\{ n,\ 2A + n + x - 3 - 2\omega,\ A + \frac{n + x - 3 - \omega}{2} \right\}.$$

Proposition 7.1. *Let i_0 be an integer as in Lemma* 5.1. *Suppose k_0 is an integer such that*

$$\lceil P_{k_0} \rceil > (n+1)^{n-1} M^n \quad \text{with } M = \max\left\{ c_2^{1/2} c_6^{-2} c_8,\ e^n \lceil P_{i_0} \rceil \right\}. \tag{7.1}$$

If P_{i-1} is irreducible and has degree n for some $i > k_0$, then

$$\left|P_{i-1}(\xi)\right|^{-1} < \lceil P_{i-1} \rceil^{\Phi(n)}. \tag{7.2}$$

If P_{i-1} is irreducible and has degree $< n$ or is reducible for some $i > k_0$, then

$$\left|P_{i-1}(\xi)\right|^{-1} < \lceil P_{i-1} \rceil^{\Phi(n-1)}. \tag{7.3}$$

We first prove two lemmas.

Lemma 7.2. *Let P be a nonzero polynomial with integer coefficients of degree $m \leqslant n$ and let*

$$\lceil P \rceil \leqslant \max\{c_6,\ e^n \lceil P_1 \rceil\}. \tag{7.4}$$

Then

$$|P(\xi)|^{-1} \leqslant c_7^{-1} \lceil P \rceil^{\Phi(m)}. \tag{7.5}$$

Proof. The result immediately follows from (7.4) by the definitions of c_7 and Φ, since

$$|P(\xi)|^{-1} \leqslant c_7^{-1} \leqslant c_7^{-1} \lceil P \rceil^{\Phi(m)}. \qquad \square$$

Lemma 7.3. *Let P be an irreducible polynomial with integer coefficients of degree $m \leqslant n$ and let*

$$\lceil P \rceil > \max\{c_6,\ e^n \lceil P_1 \rceil\}. \tag{7.6}$$

Then

$$|P(\xi)|^{-1} < \begin{cases} c_5^{-2} c_6^2 \lceil P \rceil^{\Phi(m)} & \text{if } \lceil P \rceil \leqslant M, \\ c_8^{-1} c_6^2 \lceil P \rceil^{\Phi(m)} & \text{if } \lceil P \rceil > M. \end{cases} \tag{7.7}$$

Proof. By (7.6) there exists a polynomial P_s such that

$$e^n \lceil P_s \rceil < \lceil P \rceil < e^n \lceil P_{s+1} \rceil. \tag{7.8}$$

From this, (5.5) and the definitions of δ, c_6 we deduce

$$\begin{aligned} |P_s(\xi)| &< \left(\xi^{n-1} c_3^{1-n(A-1)} c_5\right)^{-\frac{1}{A-2}} \lceil P_{s+1} \rceil^{-\omega} \\ &< \left(\xi^{n-1} c_3^{1-n(A-1)} c_5\right)^{-\frac{1}{A-2}} \left(e^{-n} \lceil P \rceil\right)^{-\omega} \\ &= c_2^{-1/2} c_6 \lceil P \rceil^{-\omega}. \end{aligned} \tag{7.9}$$

Since P is irreducible and $\lceil P \rceil > e^n \lceil P_s \rceil$, by Lemma 2.4 the polynomials P_s and P have no common root. Therefore we can apply Lemma 2.5 to them. We distinguish three cases.

Case A. Suppose (2.2)(i) or (2.3) is valid. Then by (7.8) and the definition of c_2 we have

$$\begin{aligned} 1 &< c_2^{1/2} \max\{|P_s(\xi)|, |P(\xi)|\} \max\{\lceil P_s \rceil, \lceil P \rceil\}^{n-1} \\ &= c_2^{1/2} \max\{|P_s(\xi)|, |P(\xi)|\} \lceil P \rceil^{n-1}. \end{aligned} \tag{7.10}$$

If $|P_s(\xi)| \geqslant |P(\xi)|$, then (7.9) and (7.10) yield

$$1 < c_2^{1/2} |P_s(\xi)| \lceil P \rceil^{n-1} < c_6 \lceil P \rceil^{-\omega+n-1},$$

which is $< c_6\lceil P\rceil^{-1}$, since $\omega > n$ by the definition of ω and (10.1)(iii). This is a contradiction, for $\lceil P\rceil > c_6$ by (7.6). Hence $|P_s(\xi)| < |P(\xi)|$, therefore by (7.10) and the definition of Φ we obtain

$$1 < c_2^{1/2}\left|P(\xi)\right|\lceil P\rceil^{n-1} \leqslant c_2^{1/2}\lceil P\rceil^{-1}\left|P(\xi)\right|\lceil P\rceil^{\Phi(m)}. \tag{7.11}$$

This gives (7.7). In fact, (7.11) and the definitions of c_2, c_5, c_6 imply

$$\left|P(\xi)\right|^{-1} < c_5^{-2}c_6^2\lceil P\rceil^{\Phi(m)}.$$

Similarly, using the definition of M in (7.11), we get

$$\left|P(\xi)\right|^{-1} < c_6^2c_8^{-1}\lceil P\rceil^{\Phi(m)} \quad \text{if } \lceil P\rceil > M.$$

Case B. Suppose (2.2)(ii) is valid. Then by (5.4), (7.8), (7.9) and the definition of δ we have

$$\begin{aligned}1 &< c_2 \max\left\{\left|P_s(\xi)\right|\left|P_s'(\xi)\right|\left|P'(\xi)\right|, \left|P(\xi)\right|\left|P_s'(\xi)\right|^2\right\}\lceil P_s\rceil^{m-2}\lceil P\rceil^{n-1}\\ &< c_2\delta^{-2}\left|P_s(\xi)\right|^2\left|P(\xi)\right|\lceil P\rceil^{2A+n+m-3}\\ &< c_6^2\delta^{-2}\left|P(\xi)\right|\lceil P\rceil^{2A+n+m-3-2\omega},\end{aligned}$$

where $\delta = \delta(P_s)$. Note that if $\lceil P\rceil > M$, then $\lceil P\rceil > e^n\lceil P_{i_0}\rceil > e^nH_2$ by the definition of M. From this and (7.8) it follows that $\lceil P_s\rceil > H_2$. So, $\delta(P_s) = c_8$ if $\lceil P\rceil > M$. This gives (7.7) by the definitions of δ and Φ.

Case C. Finally, suppose (2.2)(iii) is valid. Then by (5.4), (7.8), (7.9) and the definition of δ we have

$$\begin{aligned}1 &< c_2 \max\left\{\left|P(\xi)\right|\left|P_s'(\xi)\right|\left|P'(\xi)\right|, \left|P_s(\xi)\right|\left|P'(\xi)\right|^2\right\}\lceil P_s\rceil^{m-1}\lceil P\rceil^{n-2}\\ &< c_2\delta^{-2}\left|P(\xi)\right|^2\left|P_s(\xi)\right|\lceil P\rceil^{2A+n+m-3}\\ &< c_2^{1/2}c_6\delta^{-2}\left|P(\xi)\right|^2\lceil P\rceil^{2A+n+m-3-\omega},\end{aligned}$$

where $\delta = \delta(P_s)$. Using the arguments above, we deduce (7.7) from here by the definitions of c_2, c_6, δ, M and Φ. □

Proof of Proposition 7.1. Let $\deg P_{i-1} = m$, $1 \leqslant m \leqslant n$, and let $P_{i-1} = P_{i-1}^{(1)} \ldots P_{i-1}^{(\gamma)}$, $1 \leqslant \gamma \leqslant m$, where the polynomials $P_{i-1}^{(1)}, \ldots, P_{i-1}^{(\gamma)}$ have integer coefficients and are irreducible over $\mathbb{Z}$. We first note that there is an index ν with $1 \leqslant \nu \leqslant \gamma$ such that $\lceil P_{i-1}^{(\nu)}\rceil > M$. In fact, in the contrary case by the right-hand side of (2.1) we obtain

$$\lceil P_{i-1}\rceil \leqslant (n+1)^{n-1}\lceil P_{i-1}^{(1)}\rceil \cdots \lceil P_{i-1}^{(\gamma)}\rceil \leqslant (n+1)^{n-1}M^n,$$

which contradicts (7.1). From this and Lemma 7.3 it follows that

$$\left|P_{i-1}^{(\nu)}(\xi)\right|^{-1} < c_6^2c_8^{-1}\lceil P_{i-1}^{(\nu)}\rceil^{\Phi(m)} \tag{7.12}$$

for some ν with $1 \leqslant \nu \leqslant \gamma$. Combining (7.5), (7.7), (7.12) and keeping in mind the definitions of c_5, c_6, c_7, c_8, we get

$$\left|P_{i-1}(\xi)\right|^{-1} = \prod_{\nu=1}^{\gamma}\left|P_{i-1}^{(\nu)}(\xi)\right|^{-1} < c_6^2 c_8^{-1} \max\left\{c_5^{-2}c_6^2,\ c_7^{-1}\right\}^{n-1} \prod_{\nu=1}^{\gamma} \left\lceil P_{i-1}^{(\nu)} \right\rceil^{\Phi(m)},$$

which is

$$< c_6^2 c_8^{-1} \max\left\{c_5^{-2}c_6^2,\ c_7^{-1}\right\}^{n-1} e^{n\Phi(m)} \left\lceil P_{i-1} \right\rceil^{\Phi(m)} \tag{7.13}$$

by the left-hand side of (2.1). Note that $\Phi(m) < 2n-1$ by (10.1)(iii), (10.3)(iii) and the definition of Φ. From this, (5.2) and (7.13) it follows that

$$\left|P_{i-1}(\xi)\right|^{-1} < \left\lceil P_{i-1} \right\rceil^{\Phi(m)}.$$

Obviously, if P_{i-1} is irreducible and has degree n, then $m = n$, so $\Phi(m) = \Phi(n)$. Similarly, if P_{i-1} is irreducible and has degree $< n$ or is reducible, then $m \leqslant n-1$, so $\Phi(m) \leqslant \Phi(n-1)$. This gives the desired result. □

Proposition 7.4. *Let k_0 be an integer as in Proposition* 7.1. *If P_{i-1} is irreducible and has degree n for some $i > k_0$, then*

$$\left|P_{i-1}(\xi)\right|^{-1} < \left\lceil P_i \right\rceil^{\frac{2A+n-2}{3}} \left\lceil P_{i-1} \right\rceil^{\frac{n-1}{3}}. \tag{7.14}$$

Proof. From (3.1) one easily deduces that P_{i-1} and P_i have no common root. Therefore we can apply Lemma 2.5 to them. We distinguish three cases.

Case A. Suppose (2.2)(i) or (2.3) is valid. Then by (3.1)(i), (3.1)(ii) and the definitions of c_2, c_3 we have

$$1 < c_2^{1/2} \max\left\{\left|P_{i-1}(\xi)\right|, \left|P_i(\xi)\right|\right\} \max\left\{\left\lceil P_{i-1}\right\rceil, \left\lceil P_i \right\rceil\right\}^{n-1} = c_2^{1/2}\left|P_{i-1}(\xi)\right|\left\lceil P_i \right\rceil^{n-1},$$

which is $< c_2^{1/2}\left\lceil P_i \right\rceil^{-1}$ by (5.6)(ii). This contradicts (7.1).

Case B. Suppose (2.2)(ii) is valid. Then using (3.1)(i), (3.1)(ii), (5.3) and the definitions of c_2, c_3, c_8, we get

$$\begin{aligned}
1 &< c_2 \max\left\{\left|P_{i-1}(\xi)\right|\left|P'_{i-1}(\xi)\right|\left|P'_i(\xi)\right|, \left|P_i(\xi)\right|\left|P'_{i-1}(\xi)\right|^2\right\}\left\lceil P_{i-1}\right\rceil^{n-2}\left\lceil P_i \right\rceil^{n-1} \\
&< c_2 c_8^{-2}\left|P_{i-1}(\xi)\right|^2\left|P_i(\xi)\right|\left\lceil P_i \right\rceil^{A+n-1}\left\lceil P_{i-1}\right\rceil^{A+n-2} \\
&< c_2 c_3^{-1} c_8^{-2}\left|P_{i-1}(\xi)\right|^3\left\lceil P_i \right\rceil^{2A+n-2}\left\lceil P_{i-1}\right\rceil^{n-1} \\
&< \left|P_{i-1}(\xi)\right|^3\left\lceil P_i \right\rceil^{2A+n-2}\left\lceil P_{i-1}\right\rceil^{n-1}.
\end{aligned}$$

Case C. Similarly, from (2.2)(iii), (3.1)(i), (3.1)(ii), (5.3) and the definitions of c_2, c_3, c_8 we deduce

$$\begin{aligned}
1 &< c_2 \max\{|P_i(\xi)||P'_{i-1}(\xi)||P'_i(\xi)|, |P_{i-1}(\xi)||P'_i(\xi)|^2\} \lceil P_{i-1} \rceil^{n-1} \lceil P_i \rceil^{n-2} \\
&< c_2 c_8^{-2} |P_{i-1}(\xi)||P_i(\xi)|^2 \lceil P_i \rceil^{2A+n-2} \lceil P_{i-1} \rceil^{n-1} \\
&< c_2 c_3^{-2} c_8^{-2} |P_{i-1}(\xi)|^3 \lceil P_i \rceil^{2A+n-2} \lceil P_{i-1} \rceil^{n-1} \\
&< |P_{i-1}(\xi)|^3 \lceil P_i \rceil^{2A+n-2} \lceil P_{i-1} \rceil^{n-1}.
\end{aligned}$$

Clearly, Cases B and C give (7.14). □

Corollary 7.5. *Let k_0 be an integer as in Proposition* 7.1. *If P_{i-1} is irreducible and has degree n for some $i > k_0$, then for any ϱ with $0 \leqslant \varrho \leqslant 1$ we have*

$$|P_{i-1}(\xi)|^{-1} < \lceil P_i \rceil^{\frac{2A+n-2}{3}(1-\varrho)} \lceil P_{i-1} \rceil^{\frac{n-1}{3}(1-\varrho)+\Phi(n)\varrho}. \tag{7.15}$$

Proof. We raise (7.2) and (7.14) to the powers ϱ and $1-\varrho$, respectively, and multiply out the derived inequalities. □

8. Proof of (4.10)

Lemma 8.1. *Let k_0 be an integer as in Proposition* 7.1. *Suppose i and τ are integers such that $i > \tau \geqslant k_0$ and* (4.9) *holds. Then*

$$|L_{i,\tau}(\xi)| < |P_{i-1}(\xi)|^{A^2-3A+1} \lceil P_i \rceil^{(n-2)(A-1)^2} \lceil P_{i-1} \rceil^{(n-2)(A-1)}. \tag{8.1}$$

Proof. From (3.1)(ii) and (6.1)(i) it follows that for any α_1 and any nonnegative α_2 we have

$$|L_{i,\tau}(\xi)| < |P_{i-1}(\xi)|^{\frac{1}{A-1}+\alpha_1} |P_{i-1}(\xi)|^{-\alpha_1} \lceil P_i \rceil^{\alpha_2} \lceil P_{i-1} \rceil^{-n+1-\alpha_2}. \tag{8.2}$$

Put

$$\alpha_1 = -\frac{1}{A-1} + A^2 - 3A + 1. \tag{8.3}$$

Since $A > 3$ by (10.1)(iii), it follows that $\alpha_1 > 0$. We now distinguish two cases.

Case A. Suppose P_{i-1} is irreducible and has degree n. By (7.15) and (8.2) we have

$$|L_{i,\tau}(\xi)| < |P_{i-1}(\xi)|^{\frac{1}{A-1}+\alpha_1} \lceil P_i \rceil^{\frac{2A+n-2}{3}(1-\varrho)\alpha_1+\alpha_2} \lceil P_{i-1} \rceil^{(\frac{n-1}{3}(1-\varrho)+\Phi(n)\varrho)\alpha_1-n+1-\alpha_2}. \tag{8.4}$$

If $n = 3, 4, 5$, put

$$\varrho = 1 - \frac{3(n-2)(A-1)^2}{(2A+n-2)\alpha_1} \quad \text{and} \quad \alpha_2 = 0. \tag{8.5}$$

A straightforward calculation shows that $0 < \varrho < 1$. It follows from (8.3), (8.5) and (10.3)(iii) that

$$\left(\frac{n-1}{3}(1-\varrho)+\Phi(n)\varrho\right)\alpha_1-n+1=\frac{AT(A)}{(A-1)(A-2)(2A+n-2)}+(n-2)(A-1),$$

which is $(n-2)(A-1)$ by (1.4). This and (8.3)–(8.5) give (8.1). Similarly, if $n>5$, put

$$\varrho=0 \quad \text{and} \quad \alpha_2=\frac{n-1}{3}\alpha_1-n+1-(n-2)(A-1). \tag{8.6}$$

One can show (see Lemma 10.3) that

$$\alpha_2>0 \quad \text{and} \quad \frac{2A+n-2}{3}\alpha_1+\alpha_2<(n-2)(A-1)^2. \tag{8.7}$$

From (8.3), (8.4), (8.6) and (8.7) follows (8.1).

Case B. Suppose P_{i-1} is irreducible and has degree $< n$ or is reducible. By (7.3) and (8.2) we have

$$\left|L_{i,\tau}(\xi)\right| < \left|P_{i-1}(\xi)\right|^{\frac{1}{A-1}+\alpha_1}\overline{\left|P_i\right|}^{\alpha_2}\overline{\left|P_{i-1}\right|}^{\Phi(n-1)\alpha_1-n+1-\alpha_2}. \tag{8.8}$$

Put

$$\alpha_2=(n-2)(A-1)^2. \tag{8.9}$$

A straightforward calculation shows that

$$\Phi(n-1)\alpha_1-n+1-\alpha_2<(n-2)(A-1)$$

if $n=3,4,5$. Suppose $n>5$. From (8.3), (8.9) and (10.3)(ii) it follows that

$$\Phi(n-1)\alpha_1-n+1-\alpha_2=\frac{T(A)}{(A-1)(A-2)}+(n-2)(A-1),$$

which is $(n-2)(A-1)$ by (1.4). This, (8.3), (8.8) and (8.9) give (8.1). □

Corollary 8.2. *Let i and τ be integers as in Lemma* 8.1. *Then $L_{i,\tau}$ satisfies* (4.10).

Proof. If we raise both sides of (6.1)(ii) to the power $-A+1$, we obtain

$$\left|L'_{i,\tau}(\xi)\right|^{-A+1} > \left|P_{i-1}(\xi)\right|^{A^2-3A+1}\overline{\left|P_i\right|}^{(n-2)(A-1)^2}\overline{\left|P_{i-1}\right|}^{(n-2)(A-1)}.$$

Combining this with (8.1), we get (4.10). □

9. Proof of Theorem

Lemma 9.1. *Let i and τ be integers as in Lemma* 8.1. *Then*

$$\left|P_{\tau-1}(\xi)\right|\prod_{\nu=0}^{n-2}\left\lceil Q_{\tau}^{(\nu)}\right\rceil \leqslant (2n)^{n/2}c_3^n\left|P_{i-1}(\xi)\right|\prod_{\nu=0}^{n-2}\left\lceil Q_{i}^{(\nu)}\right\rceil.$$

Proof. Suppose to the contrary

$$\left|P_{\tau-1}(\xi)\right|\prod_{\nu=0}^{n-2}\left\lceil Q_{\tau}^{(\nu)}\right\rceil > (2n)^{n/2}c_3^n\left|P_{i-1}(\xi)\right|\prod_{\nu=0}^{n-2}\left\lceil Q_{i}^{(\nu)}\right\rceil. \tag{9.1}$$

By (4.1) and (4.8) we have

$$\left|L_{i,\tau}(\xi)\right| \leqslant (2n)^{n/2}c_3^{n-1}\left|P_{i-1}(\xi)\right|\prod_{\nu=0}^{n-2}\left\lceil Q_{i}^{(\nu)}\right\rceil\prod_{\nu=0}^{n-2}\left\lceil Q_{\tau}^{(\nu)}\right\rceil^{-1},$$

which is $< c_3^{-1}|P_{\tau-1}(\xi)|$ by (9.1). This, (4.1) and (4.8) imply

$$\begin{aligned} &\text{(i)} \quad \left|L_{i,\tau}(\xi)\right| < c_3^{-1}\left|P_{\tau-1}(\xi)\right|,\\ &\text{(ii)} \quad |d_{\kappa_\nu}| \leqslant c_3^{-1}\left\lceil Q_{\tau}^{(\nu-1)}\right\rceil \quad (\nu = 1,\dots,n-1), \end{aligned} \tag{9.2}$$

where $d_{\kappa_1},\dots,d_{\kappa_{n-1}}$ are coefficients of $L_{i,\tau}$. From (9.2) by the minimality property of $Q_{\tau}^{(n-2)}$ we get

$$\left\lceil L_{i,\tau}\right\rceil \geqslant \left\lceil Q_{\tau}^{(n-2)}\right\rceil. \tag{9.3}$$

This, (3.6), (9.2)(ii) and the definition of c_3 give

$$|d_{\kappa_\nu}| \leqslant c_3^{-1}\left\lceil Q_{\tau}^{(\nu-1)}\right\rceil \leqslant c_3^{-1}\left\lceil Q_{\tau}^{(n-2)}\right\rceil \leqslant c_3^{-1}\left\lceil L_{i,\tau}\right\rceil \leqslant \xi^{n-1}\left\lceil L_{i,\tau}\right\rceil \quad (\nu = 1,\dots,n-1).$$

Therefore $L_{i,\tau}$ satisfies the conditions of Lemma 2.2, by which

$$\left|L'_{i,\tau}(\xi)\right| > \xi^{n-1}\left\lceil L_{i,\tau}\right\rceil. \tag{9.4}$$

We also note that by (3.6) and (9.3) we have $\lceil L_{i,\tau}\rceil \geqslant \lceil P_\tau\rceil$, which is $> H_2$, since $\tau \geqslant k_0$. Therefore we can apply (5.3) to $L_{i,\tau}$. From this and (9.4) it follows that

$$\left|L_{i,\tau}(\xi)\right| > c_8\left|L'_{i,\tau}(\xi)\right|\left\lceil L_{i,\tau}\right\rceil^{-A} > c_8\xi^{(n-1)A}\left|L'_{i,\tau}(\xi)\right|^{-A+1},$$

which is $> |L'_{i,\tau}(\xi)|^{-A+1}$ by (5.2). We obtain a contradiction with (4.10). □

Proof of Theorem. Choose an increasing sequence of integers $\{m_t\}$ such that $k_0 = m_1 < m_2 < \cdots$ and

$$\lceil P_{m_{t+1}-1} \rceil \leqslant c_4 \lceil P_{m_t} \rceil < \lceil P_{m_{t+1}} \rceil, \quad t = 1, 2, \ldots. \tag{9.5}$$

By Lemma 9.1 we have

$$\left|P_{m_t-1}(\xi)\right| \prod_{\nu=0}^{n-2} \left\lceil Q_{m_t}^{(\nu)} \right\rceil \leqslant (2n)^{n/2} c_3^n \left|P_{m_{t+1}-1}(\xi)\right| \prod_{\nu=0}^{n-2} \left\lceil Q_{m_{t+1}}^{(\nu)} \right\rceil, \quad t = 1, 2, \ldots.$$

Let ℓ be some integer $\geqslant 1$. If we multiply these inequalities together for all t with $1 \leqslant t \leqslant \ell$, we obtain

$$\left|P_{m_1-1}(\xi)\right| \prod_{\nu=0}^{n-2} \left\lceil Q_{m_1}^{(\nu)} \right\rceil \leqslant (2n)^{n\ell/2} c_3^{n\ell} \left|P_{m_{\ell+1}-1}(\xi)\right| \prod_{\nu=0}^{n-2} \left\lceil Q_{m_{\ell+1}}^{(\nu)} \right\rceil,$$

hence

$$\left|P_{m_1-1}(\xi)\right| < (2n)^{n\ell/2} c_3^{n\ell} \left|P_{m_{\ell+1}-1}(\xi)\right| \prod_{\nu=0}^{n-2} \left\lceil Q_{m_{\ell+1}}^{(\nu)} \right\rceil. \tag{9.6}$$

Substituting (5.6)(iii) into (9.6), using the definitions of c_3, c_4 and then applying (5.6)(i), we get

$$\left|P_{m_1-1}(\xi)\right| < (2n)^{n\ell/2} c_3^{n\ell} \left|P_{m_{\ell+1}-1}(\xi)\right|^{\frac{1}{A-1}} < (2n)^{n\ell/2} c_3^{n\ell} \lceil P_{m_{\ell+1}} \rceil^{-\frac{\omega}{A-1}}. \tag{9.7}$$

By the right-hand side of (9.5) we have

$$\lceil P_{m_{t+1}} \rceil^{-1} < c_4^{-1} \lceil P_{m_t} \rceil^{-1}, \quad t = 1, 2, \ldots.$$

If we multiply these inequalities together for all t with $1 \leqslant t \leqslant \ell$, we obtain

$$\lceil P_{m_{\ell+1}} \rceil^{-1} < c_4^{-\ell} \lceil P_{m_1} \rceil^{-1} < c_4^{-\ell}.$$

Using this in (9.7) and keeping in mind the definitions of c_4, ω, we get

$$\left|P_{m_1-1}(\xi)\right| < (2n)^{n\ell/2} c_3^{n\ell} c_4^{-\frac{\omega}{A-1}\ell} = 2^{-\ell}.$$

Letting $\ell \to \infty$, we come to a contradiction. Thus, the assumption (1.5) can not be true. This completes the proof of the theorem. □

10. Calculations concerning the exponent A

Lemma 10.1. *We have*

$$\begin{aligned} &\text{(i)} && \lim_{n\to\infty}(A-n/2)=4,\\ &\text{(ii)} && \frac{n}{2}+3.5<A<\frac{n}{2}+4 \quad \text{for } n\geqslant 26,\\ &\text{(iii)} && 3<A<n+1 \quad \text{for } n\geqslant 3. \end{aligned} \tag{10.1}$$

Proof. Consider $T(x)$ from (1.4). To prove (10.1)(i), we note that if $n>10$ and $x>n/2$, then

$$\left|(2n-41)x^2-(3n-29)x+2n-10\right|<7nx^2<14x^3,$$

so

$$T(x)\begin{cases} < x^3(2x^2-(n+12)x+2n+44),\\ > x^3(2x^2-(n+12)x+2n+16). \end{cases}$$

Of the quadratic factors the second splits into $(x-2)(2x-n-8)$ and the first has real roots if $n>10$. Consequently,

$$B<A<\frac{n}{2}+4, \tag{10.2}$$

where $B=B(n)$ is the largest root of $2x^2-(n+12)x+2n+44$. Since

$$B=\frac{n}{2}+4+\mathrm{O}\left(\frac{1}{n}\right),$$

the same follows for A. This gives (10.1)(i).

We now prove (10.1)(ii). It is easy to verify that $B\geqslant n/2+3.5$ if $n\geqslant 53$. A straightforward calculation also shows that $A>n/2+3.5$ if $26\leqslant n<53$. This and (10.2) give (10.1)(ii).

Finally, (10.1)(iii) follows from (10.1)(ii) if $n\geqslant 26$. One can check that it is also true if $3\leqslant n<26$. □

Lemma 10.2. *We have*

$$\begin{aligned} &\text{(i)} && \Phi(2)=\Phi(2,n)=A+1-\frac{\omega}{2} \quad \text{for } n=3,\\ &\text{(ii)} && \Phi(n-1)=\Phi(n-1,n)=2A+2n-4-2\omega \quad \text{for } n\geqslant 4,\\ &\text{(iii)} && \Phi(n)=\Phi(n,n)=2A+2n-3-2\omega \quad \text{for } n\geqslant 3. \end{aligned} \tag{10.3}$$

Proof. One can check (10.3)(i) directly. To prove (10.3)(ii) and (10.3)(iii), we show that

$$2A+n+x-3-2\omega>\max\left\{n,\ A+\frac{n+x-3-\omega}{2}\right\}$$

for $x=n-1$ and $x=n$. Obviously, we only need to prove it for $x=n-1$, i.e.,

$$2A+2n-4-2\omega > \max\left\{n,\ A+n-2-\frac{\omega}{2}\right\}.$$

We first show that

$$2A+2n-4-2\omega > A+n-2-\frac{\omega}{2},$$

which can be rewritten as

$$\frac{2A^2-(n+5)A-n+5}{2(A-2)} > 0. \tag{10.4}$$

It is easy to verify that (10.4) is true if $3 \leqslant n < 26$. Suppose $n \geqslant 26$. The function

$$f(x)=2x^2-(n+5)x-n+5$$

is increasing for $x \geqslant (n+5)/4$. Since $A > n/2+3.5 > (n+5)/4$ by (10.1)(ii), we obtain

$$f(A) > f\left(\frac{n}{2}+3.5\right)=12,$$

which gives (10.4).

To show that

$$A+n-2-\frac{\omega}{2} > n,$$

we rewrite it as

$$\frac{2A^2-(n+7)A+n+7}{2(A-2)} > 0,$$

which is true by (10.1)(iii) and (10.4). □

Lemma 10.3. *The estimates* (8.7) *hold for any* $n > 5$.

Proof. By (8.3) and (8.6) we have

$$\begin{aligned}\alpha_2 &= \frac{n-1}{3}\alpha_1 - n + 1 - (n-2)(A-1)\\ &= \frac{(n-1)A^3-(7n-10)A^2+(7n-13)A-2n+5}{3(A-1)}.\end{aligned}$$

One can check that $\alpha_2 > 0$ if $5 < n < 26$. Suppose $n \geqslant 26$. Then by (10.1)(ii) we get

$$(n-1)A^3-(7n-10)A^2 > (n-1)\frac{n}{2}A^2-(7n-10)A^2 = A^2\left(\frac{n^2}{2}-\frac{15n}{2}+10\right),$$

which is positive. Also, it is easy to see that $(7n-13)A-2n+5>0$. Therefore $\alpha_2>0$.

To prove the second estimate from (8.7), we note that by (8.3) and (8.6) we obtain

$$\frac{2A+n-2}{3}\alpha_1+\alpha_2 = \frac{2A^4-(n+5)A^3-(2n-8)A^2+(2n-7)A-n+3}{3(A-1)}+(n-2)(A-1)^2. \tag{10.5}$$

We now show that

$$2A^4-(n+5)A^3-(2n-8)A^2+(2n-7)A-n+3<0. \tag{10.6}$$

It is easy to verify that (10.6) is true if $5<n<48$. Suppose $n\geqslant 48$. Note that $(2n-7)A-n+3<4A^2$, since $A>n/2$ by (10.1)(ii). Hence

$$2A^4-(n+5)A^3-(2n-8)A^2+(2n-7)A-n+3<A^2\bigl(2A^2-(n+5)A-2n+12\bigr).$$

The function

$$f(x)=2x^2-(n+5)x-2n+12$$

is increasing for $x>(n+5)/4$. Since $A>n/2+3.5>(n+5)/4$ and $A<n/2+4$ by (10.1)(ii), we get

$$f(A)<f\left(\frac{n}{2}+4\right)=-\frac{n}{2}+24\leqslant 0,$$

which gives (10.6). Clearly, (10.5) and (10.6) imply the second estimate from (8.7). □

Acknowledgments

This paper was written in the Institute of Mathematics at the University of Ulm. I would like to thank the faculty and staff of this institute for their warm hospitality during the whole period of my stay. I am particularly grateful to E. Wirsing for helpful comments and constant attention to my work. Gratitude also goes to J. Vaaler for enlightening discussions of my paper. Finally, I want to acknowledge the outstanding contribution of the anonymous referee to the revision of this paper.

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