On approximation of real numbers by algebraic numbers of bounded degree

BY

K. I. Tsishchanka

Theorem 1 (Dirichlet, 1842): **For any real irrational number ξ there exist infinitely many rational numbers p/q such that**

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Theorem 1′: **For any real irrational number ξ there exist infinitely many polynomials $P(x) = ax + b$ with integer coefficients such that**

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Theorem 2: **For any real number $\xi \notin A_n$ there exist infinitely many polynomials $P(x) \in Z[x]$ of degree $\leq n$ such that**

$$|P(\xi)| < c(\xi, n)\lceil P\rceil^{-n}.$$

$$
\begin{array}{ccc}
\left|\xi - \dfrac{p}{q}\right| < q^{-2} & \longrightarrow & |q\xi - p| < q^{-1} \\
 & & \big\downarrow \\
|\xi - \alpha| \ll H(\alpha)^{-n-1} & \longleftarrow & |P(\xi)| \ll \lceil P \rceil^{-n}
\end{array}
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$n = 1 \;\Rightarrow\; |\xi - \alpha| \ll H(\alpha)^{-2}$ **(Dirichlet,1842)**

$n = 2 \;\Rightarrow\; |\xi - \alpha| \ll H(\alpha)^{-3}$ **(Davenport – Schmidt, 1967)**

$n \geq 3 \;\Rightarrow\; |\xi - \alpha| \ll H(\alpha)^{-\frac{n}{2}-\frac{3}{2}}$ **(Wirsing, 1961)**

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LEMMA 1: **We have $|\xi - \alpha| \ll \dfrac{|P(\xi)|}{|P'(\xi)|}$ where α is the root of P closest to ξ.**

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Lemma 1: **We have $|\xi - \alpha| \ll \dfrac{|P(\xi)|}{|P'(\xi)|}$ where α is the root of P closest to ξ.**

Lemma 2: **There are infinitely many polynomials P, $Q \in Z[x]$ of degree $\leq n$, such that**

$$|P(\xi)| \ll \overline{|P|}^{-n}$$
$$|Q(\xi)| \ll \overline{|P|}^{-n}$$
$$\overline{|Q|} \ll \overline{|P|}$$

and

P, Q have no common root

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Lemma 1: We have $|\xi - \alpha| \ll \dfrac{|P(\xi)|}{|P'(\xi)|}$ where α is the root of P closest to ξ.

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$$|P(\xi)| \ll \lceil P \rceil^{-n}$$
$$|Q(\xi)| \ll \lceil P \rceil^{-n}$$
$$\lceil Q \rceil \ll \lceil P \rceil$$

and

P, Q have no common root

Lemma 3: Let $P, Q \in Z[x]$ be polynomials of degree d with $1 < d \leq n$. Suppose that P and Q have no common root. Then at least one of the following estimates is true:

$$1 \ll \max\{|P(\xi)|, |Q(\xi)|\}^2 \max\left\{\lceil P \rceil, \lceil Q \rceil\right\}^{2n-2}$$
$$1 \ll \max\left\{|P(\xi)||P'(\xi)||Q'(\xi)|, |Q(\xi)||P'(\xi)|^2\right\} \lceil P \rceil^{n-2} \lceil Q \rceil^{n-1}$$
$$1 \ll \max\left\{|Q(\xi)||P'(\xi)||Q'(\xi)|, |P(\xi)||Q'(\xi)|^2\right\} \lceil P \rceil^{n-1} \lceil Q \rceil^{n-2}$$

PROOF: **Let**

$$\boldsymbol{P(x) = a_\ell x^\ell + \ldots + a_1 x + a_0 = a_\ell (x - \alpha_1) \ldots (x - \alpha_\ell)}$$
$$\boldsymbol{Q(x) = b_m x^m + \ldots + b_1 x + b_0 = b_m (x - \beta_1) \ldots (x - \beta_m)}$$

Consider

$$\boldsymbol{a_\ell^m b_m^\ell \prod_{\substack{1 \le i \le \ell \\ 1 \le j \le m}} (\alpha_i - \beta_j)}$$

Proof: **Let**

$$P(x) = a_\ell x^\ell + \ldots + a_1 x + a_0 = a_\ell (x - \alpha_1) \ldots (x - \alpha_\ell)$$
$$Q(x) = b_m x^m + \ldots + b_1 x + b_0 = b_m (x - \beta_1) \ldots (x - \beta_m)$$

Consider

$$R(P, Q) = a_\ell^m b_m^\ell \prod_{\substack{1 \le i \le \ell \\ 1 \le j \le m}} (\alpha_i - \beta_j)$$

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We have

$$R(P, Q) = a_\ell^m b_m^\ell \prod_{\substack{1 \le i \le \ell \\ 1 \le j \le m}} (\alpha_i - \beta_j) \neq 0$$

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We have

$$|R(P, Q)| = |a_\ell^m b_m^\ell \prod_{\substack{1 \le i \le \ell \\ 1 \le j \le m}} (\alpha_i - \beta_j)| > 0$$

PROOF: **Let**

$$\begin{aligned} P(x) &= a_\ell x^\ell + \ldots + a_1 x + a_0 = a_\ell (x-\alpha_1)\ldots(x-\alpha_\ell) \\ Q(x) &= b_m x^m + \ldots + b_1 x + b_0 = b_m (x-\beta_1)\ldots(x-\beta_m) \end{aligned}$$

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$$R(P,Q) = \left| \begin{array}{cccccc} a_\ell & \ldots & a_1 & a_0 & & \\ \ddots & & \ddots & \ddots & & \\ & & a_\ell & \ldots & a_1 & a_0 \\ b_m & \ldots & b_1 & b_0 & & \\ \ddots & & \ddots & \ddots & & \\ & & b_m & \ldots & b_1 & b_0 \end{array} \right| \begin{array}{l} \left.\vphantom{\begin{array}{c} a \\ a \\ a \end{array}}\right\} m \\ \left.\vphantom{\begin{array}{c} a \\ a \\ a \end{array}}\right\} \ell \end{array}$$

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$$1 \ll \max\{|P(\xi)|, |Q(\xi)|\}^2 \max\{\lceil P\rceil, \lceil Q\rceil\}^{2n-2}$$

$$1 \ll \max\{|P(\xi)||P'(\xi)||Q'(\xi)|, |Q(\xi)||P'(\xi)|^2\} \lceil P\rceil^{n-2}\lceil Q\rceil^{n-1}$$

$$1 \ll \max\{|Q(\xi)||P'(\xi)||Q'(\xi)|, |P(\xi)||Q'(\xi)|^2\} \lceil P\rceil^{n-1}\lceil Q\rceil^{n-2}$$

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$$\begin{vmatrix} a_\ell & \ldots & a_1 & a_0 & \\ \ddots & & \ddots & \ddots & \\ & a_\ell & \ldots & a_1 & a_0 \\ b_m & \ldots & b_1 & b_0 & \\ \ddots & & \ddots & \ddots & \\ & b_m & \ldots & b_1 & b_0 \end{vmatrix}$$

<u>Proof</u>: **Let**

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$$\ll \max\left\{|P(\xi)||P'(\xi)||Q'(\xi)|, |Q(\xi)||P'(\xi)|^2\right\} \overline{|P|}^{n-2}\overline{|Q|}^{n-1}$$

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$$\ll \max\left\{|Q(\xi)||P'(\xi)||Q'(\xi)|, |P(\xi)||Q'(\xi)|^2\right\} \overline{|P|}^{n-1}\overline{|Q|}^{n-2}$$

$$
\begin{vmatrix}
\frac{P^{(5)}(\xi)}{5!} & \frac{P^{(4)}(\xi)}{4!} & \frac{P^{(3)}(\xi)}{3!} & \frac{P''(\xi)}{2!} & P'(\xi) & P(\xi) & 0 & 0 & 0 & 0 \\
0 & \frac{P^{(5)}(\xi)}{5!} & \frac{P^{(4)}(\xi)}{4!} & \frac{P^{(3)}(\xi)}{3!} & \frac{P''(\xi)}{2!} & P'(\xi) & P(\xi) & 0 & 0 & 0 \\
0 & 0 & \frac{P^{(5)}(\xi)}{5!} & \frac{P^{(4)}(\xi)}{4!} & \frac{P^{(3)}(\xi)}{3!} & \frac{P''(\xi)}{2!} & P'(\xi) & P(\xi) & 0 & 0 \\
0 & 0 & 0 & \frac{P^{(5)}(\xi)}{5!} & \frac{P^{(4)}(\xi)}{4!} & \frac{P^{(3)}(\xi)}{3!} & \frac{P''(\xi)}{2!} & P'(\xi) & P(\xi) & 0 \\
0 & 0 & 0 & 0 & \frac{P^{(5)}(\xi)}{5!} & \frac{P^{(4)}(\xi)}{4!} & \frac{P^{(3)}(\xi)}{3!} & \frac{P''(\xi)}{2!} & P'(\xi) & P(\xi) \\
\frac{Q^{(5)}(\xi)}{5!} & \frac{Q^{(4)}(\xi)}{4!} & \frac{Q^{(3)}(\xi)}{3!} & \frac{Q''(\xi)}{2!} & Q'(\xi) & Q(\xi) & 0 & 0 & 0 & 0 \\
0 & \frac{Q^{(5)}(\xi)}{5!} & \frac{Q^{(4)}(\xi)}{4!} & \frac{Q^{(3)}(\xi)}{3!} & \frac{Q''(\xi)}{2!} & Q'(\xi) & Q(\xi) & 0 & 0 & 0 \\
0 & 0 & \frac{Q^{(5)}(\xi)}{5!} & \frac{Q^{(4)}(\xi)}{4!} & \frac{Q^{(3)}(\xi)}{3!} & \frac{Q''(\xi)}{2!} & Q'(\xi) & Q(\xi) & 0 & 0 \\
0 & 0 & 0 & \frac{Q^{(5)}(\xi)}{5!} & \frac{Q^{(4)}(\xi)}{4!} & \frac{Q^{(3)}(\xi)}{3!} & \frac{Q''(\xi)}{2!} & Q'(\xi) & Q(\xi) & 0 \\
0 & 0 & 0 & 0 & \frac{Q^{(5)}(\xi)}{5!} & \frac{Q^{(4)}(\xi)}{4!} & \frac{Q^{(3)}(\xi)}{3!} & \frac{Q''(\xi)}{2!} & Q'(\xi) & Q(\xi)
\end{vmatrix}
$$

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\left|\begin{array}{ccccccc|ccc}
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0 & \frac{P^{(5)}(\xi)}{5!} & \frac{P^{(4)}(\xi)}{4!} & \frac{P^{(3)}(\xi)}{3!} & \frac{P''(\xi)}{2!} & P'(\xi) & P(\xi) & 0 & 0 & 0 \\
0 & 0 & \frac{P^{(5)}(\xi)}{5!} & \frac{P^{(4)}(\xi)}{4!} & \frac{P^{(3)}(\xi)}{3!} & \frac{P''(\xi)}{2!} & P'(\xi) & P(\xi) & 0 & 0 \\
0 & 0 & 0 & \frac{P^{(5)}(\xi)}{5!} & \frac{P^{(4)}(\xi)}{4!} & \frac{P^{(3)}(\xi)}{3!} & \frac{P''(\xi)}{2!} & P'(\xi) & P(\xi) & 0 \\
0 & 0 & 0 & 0 & \frac{P^{(5)}(\xi)}{5!} & \frac{P^{(4)}(\xi)}{4!} & \frac{P^{(3)}(\xi)}{3!} & \frac{P''(\xi)}{2!} & P'(\xi) & P(\xi) \\
\frac{Q^{(5)}(\xi)}{5!} & \frac{Q^{(4)}(\xi)}{4!} & \frac{Q^{(3)}(\xi)}{3!} & \frac{Q''(\xi)}{2!} & Q'(\xi) & Q(\xi) & 0 & 0 & 0 & 0 \\
0 & \frac{Q^{(5)}(\xi)}{5!} & \frac{Q^{(4)}(\xi)}{4!} & \frac{Q^{(3)}(\xi)}{3!} & \frac{Q''(\xi)}{2!} & Q'(\xi) & Q(\xi) & 0 & 0 & 0 \\
0 & 0 & \frac{Q^{(5)}(\xi)}{5!} & \frac{Q^{(4)}(\xi)}{4!} & \frac{Q^{(3)}(\xi)}{3!} & \frac{Q''(\xi)}{2!} & Q'(\xi) & Q(\xi) & 0 & 0 \\
0 & 0 & 0 & \frac{Q^{(5)}(\xi)}{5!} & \frac{Q^{(4)}(\xi)}{4!} & \frac{Q^{(3)}(\xi)}{3!} & \frac{Q''(\xi)}{2!} & Q'(\xi) & Q(\xi) & 0 \\
0 & 0 & 0 & 0 & \frac{Q^{(5)}(\xi)}{5!} & \frac{Q^{(4)}(\xi)}{4!} & \frac{Q^{(3)}(\xi)}{3!} & \frac{Q''(\xi)}{2!} & Q'(\xi) & Q(\xi)
\end{array}\right|
$$

$$
\left|
\begin{array}{ccccccc|ccc}
\boxed{P} & \boxed{P} & \boxed{P} & \boxed{P} & \boxed{P} & \boxed{P} & 0 & 0 & 0 & 0 \\
0 & \boxed{P} & \boxed{P} & \boxed{P} & \boxed{P} & \boxed{P} & \boxed{P} & 0 & 0 & 0 \\
0 & 0 & \boxed{P} & \boxed{P} & \boxed{P} & \boxed{P} & \boxed{P} & P(\xi) & 0 & 0 \\
0 & 0 & 0 & \boxed{P} & \boxed{P} & \boxed{P} & \boxed{P} & P'(\xi) & P(\xi) & 0 \\
0 & 0 & 0 & 0 & \boxed{P} & \boxed{P} & \boxed{P} & \frac{P''(\xi)}{2!} & P'(\xi) & P(\xi) \\
\boxed{Q} & \boxed{Q} & \boxed{Q} & \boxed{Q} & \boxed{Q} & \boxed{Q} & 0 & 0 & 0 & 0 \\
0 & \boxed{Q} & \boxed{Q} & \boxed{Q} & \boxed{Q} & \boxed{Q} & \boxed{Q} & 0 & 0 & 0 \\
0 & 0 & \boxed{Q} & \boxed{Q} & \boxed{Q} & \boxed{Q} & \boxed{Q} & Q(\xi) & 0 & 0 \\
0 & 0 & 0 & \boxed{Q} & \boxed{Q} & \boxed{Q} & \boxed{Q} & Q'(\xi) & Q(\xi) & 0 \\
0 & 0 & 0 & 0 & \boxed{Q} & \boxed{Q} & \boxed{Q} & \frac{Q''(\xi)}{2!} & Q'(\xi) & Q(\xi)
\end{array}
\right|
$$

$$
\left|\begin{array}{ccccccc|ccc}
\boxed{P} & \boxed{P} & \boxed{P} & \boxed{P} & \boxed{P} & \boxed{P} & 0 & 0 & 0 & 0 \\
0 & \boxed{P} & \boxed{P} & \boxed{P} & \boxed{P} & \boxed{P} & \boxed{P} & 0 & 0 & 0 \\
0 & 0 & \boxed{P} & \boxed{P} & \boxed{P} & \boxed{P} & \boxed{P} & P(\xi) & 0 & 0 \\
0 & 0 & 0 & \boxed{P} & \boxed{P} & \boxed{P} & \boxed{P} & P'(\xi) & P(\xi) & 0 \\
0 & 0 & 0 & 0 & \boxed{P} & \boxed{P} & \boxed{P} & \boxed{P} & P'(\xi) & P(\xi) \\
\boxed{Q} & \boxed{Q} & \boxed{Q} & \boxed{Q} & \boxed{Q} & \boxed{Q} & 0 & 0 & 0 & 0 \\
0 & \boxed{Q} & \boxed{Q} & \boxed{Q} & \boxed{Q} & \boxed{Q} & \boxed{Q} & 0 & 0 & 0 \\
0 & 0 & \boxed{Q} & \boxed{Q} & \boxed{Q} & \boxed{Q} & \boxed{Q} & Q(\xi) & 0 & 0 \\
0 & 0 & 0 & \boxed{Q} & \boxed{Q} & \boxed{Q} & \boxed{Q} & Q'(\xi) & Q(\xi) & 0 \\
0 & 0 & 0 & 0 & \boxed{Q} & \boxed{Q} & \boxed{Q} & \boxed{Q} & Q'(\xi) & Q(\xi)
\end{array}\right|
$$

$$
\left|\begin{array}{ccccccc:ccc}
\boxed{P} & \boxed{P} & \boxed{P} & \boxed{P} & \boxed{P} & \boxed{P} & 0 & 0 & 0 & 0 \\
0 & \boxed{P} & \boxed{P} & \boxed{P} & \boxed{P} & \boxed{P} & \boxed{P} & 0 & 0 & 0 \\
0 & 0 & \boxed{P} & \boxed{P} & \boxed{P} & \boxed{P} & \boxed{P} & \boxed{P} & 0 & 0 \\
0 & 0 & 0 & \boxed{P} & \boxed{P} & \boxed{P} & \boxed{P} & P'(\xi) & P(\xi) & 0 \\
0 & 0 & 0 & 0 & \boxed{P} & \boxed{P} & \boxed{P} & \boxed{P} & P'(\xi) & P(\xi) \\
\boxed{Q} & \boxed{Q} & \boxed{Q} & \boxed{Q} & \boxed{Q} & \boxed{Q} & 0 & 0 & 0 & 0 \\
0 & \boxed{Q} & \boxed{Q} & \boxed{Q} & \boxed{Q} & \boxed{Q} & \boxed{Q} & 0 & 0 & 0 \\
0 & 0 & \boxed{Q} & \boxed{Q} & \boxed{Q} & \boxed{Q} & \boxed{Q} & \boxed{Q} & 0 & 0 \\
0 & 0 & 0 & \boxed{Q} & \boxed{Q} & \boxed{Q} & \boxed{Q} & Q'(\xi) & Q(\xi) & 0 \\
0 & 0 & 0 & 0 & \boxed{Q} & \boxed{Q} & \boxed{Q} & \boxed{Q} & Q'(\xi) & Q(\xi)
\end{array}\right|
$$

$$
\left\|
\begin{array}{ccccccc:ccc}
\overline{|P|} & \overline{|P|} & \overline{|P|} & \overline{|P|} & \overline{|P|} & \overline{|P|} & 0 & 0 & 0 & 0 \\
0 & \overline{|P|} & \overline{|P|} & \overline{|P|} & \overline{|P|} & \overline{|P|} & \overline{|P|} & 0 & 0 & 0 \\
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\end{array}
\right\|
$$

$$
\ll \max\{|P(\xi)|, |Q(\xi)|\}^2 \max\left\{\overline{|P|}, \overline{|Q|}\right\}^8
$$

$$
\left\|\begin{array}{ccccccc|ccc}
\overline{|P|} & \overline{|P|} & \overline{|P|} & \overline{|P|} & \overline{|P|} & \overline{|P|} & 0 & 0 & 0 & 0 \\
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$$

$$
\ll \max\left\{|P(\xi)||P'(\xi)||Q'(\xi)|, |Q(\xi)||P'(\xi)|^2\right\} \overline{|P|}^3 \overline{|Q|}^4
$$

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$$|\xi - \alpha| \ll H(\alpha)^{-\frac{n}{2}-\frac{3}{2}}$$

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n	Th. 3, 1961	Th. 4, 1961	Th. 5, 1993	Th. 6, 2007	Conjecture
3	3	3.28	3.5	3.73	4
4	3.5	3.82	4.12	4.45	5
5	4	4.35	4.71	5.14	6
6	4.5	4.87	5.28	5.76	7
7	5	5.39	5.84	6.36	8
8	5.5	5.9	6.39	6.93	9
9	6	6.41	6.93	7.50	10
10	6.5	6.92	7.47	8.06	11
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Proof: **For any $H > 0$ there exist**

$$Q(x) = a_n x^n + \ldots + a_1 x + a_0 \in \mathbb{Z}[x]$$

such that

$$|Q(\xi)| \ll H^{-n}, \quad |a_n| < H, \ \ldots \quad |a_2| < H, \quad |a_1| < H$$

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$$1 \ll \max\{|P(\xi)|, |Q(\xi)|\}^2 \max\left\{\lceil P \rceil, \lceil Q \rceil\right\}^{2n-2}$$

$$1 \ll \max\left\{|P(\xi)||P'(\xi)||Q'(\xi)|, |Q(\xi)||P'(\xi)|^2\right\} \lceil P \rceil^{n-2} \lceil Q \rceil^{n-1}$$

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If $\boxed{Q} \gg H$, then $|Q'(\xi)| \gg H$ and we apply

$$|\xi - \alpha| \ll \frac{|Q(\xi)|}{|Q'(\xi)|}.$$

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Proof: For any $H > 0$ there exist

$$Q(x) = a_n x^n + \ldots + a_1 x + a_0 \in \mathbb{Z}[x]$$

such that

$$|Q(\xi)| \ll H^{-n-2\epsilon}, \quad |a_n| < H, \; \ldots \quad |a_2| < H^{1+\epsilon}, \quad |a_1| < H^{1+\epsilon}$$

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IDEA: Let $n = 3$. Consider three polynomials

$$Q(x),\ P_i(x),\ P_{i-1}(x) \in \mathbb{Z}[x]$$

such that

$$|Q(\xi)| \ll H_i^{-3-2\epsilon}, \quad |c_3| \ll H_i, \quad |c_2| \ll H_i^{1+\epsilon}, \quad |c_1| \ll H_i^{1+\epsilon}$$

$$|P_i(\xi)| \ll H_i^{-3}, \quad |b_3| \ll H_i, \quad |b_2| \ll H_i, \quad |b_1| \ll H_i$$

$$|P_{i-1}(\xi)| \ll H_{i-1}^{-3}, \quad |a_3| \ll H_{i-1}, \quad |a_2| \ll H_{i-1}, \quad |a_1| \ll H_{i-1}$$

We construct

$$L(x) = Q(x)a + P_i(x)b + P_{i-1}(x)c$$

such that

$$|L(\xi)| \ll H_{i-1}^{-3-2\epsilon}, \quad |d_3| \ll H_{i-1}, \quad |d_2| \ll H_{i-1}, \quad |d_1| \ll H_{i-1}^{1+\epsilon}$$

3. Construction of auxiliary polynomials

Put

$$c_3 = \max\left(\xi^{-n+1},\ n!\right).$$

We first construct a sequence of polynomials P_i such that

$$\begin{aligned}
&\text{(i)} \quad \tfrac{1}{2} > |P_1(\xi)| > c_3|P_2(\xi)| > \ldots > c_3^{i-1}|P_i(\xi)| > \ldots, \\
&\text{(ii)} \quad \overline{|P_1|} < \overline{|P_2|} < \ldots < \overline{|P_i|} < \ldots, \\
&\text{(iii)} \quad \text{for any polynomial } P \text{ with } \overline{|P|} < \overline{|P_{i+1}|} \text{ we have } |P(\xi)| \geq c_3^{-1}|P_i(\xi)|.
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\end{aligned} \tag{3.1}$$

<u>Example</u>: Let $n = 1$ and $\xi = \dfrac{1+\sqrt{5}}{2}$. Then

$P_1(x) = x - 2$	$\lvert P_1(\xi)\rvert \approx 0.3819$	$\boxed{P_1} = 2$
$P_2(x) = 2x - 3$	$\lvert P_2(\xi)\rvert \approx 0.2361$	$\boxed{P_2} = 3$
$P_3(x) = 3x - 5$	$\lvert P_3(\xi)\rvert \approx 0.1459$	$\boxed{P_3} = 5$
$P_4(x) = 5x - 8$	$\lvert P_4(\xi)\rvert \approx 0.0902$	$\boxed{P_4} = 8$
$P_5(x) = 8x - 13$	$\lvert P_5(\xi)\rvert \approx 0.0557$	$\boxed{P_5} = 13$
$P_6(x) = 13x - 21$	$\lvert P_6(\xi)\rvert \approx 0.0344$	$\boxed{P_6} = 21$
$P_7(x) = 21x - 34$	$\lvert P_7(\xi)\rvert \approx 0.0213$	$\boxed{P_7} = 34$

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&\text{(iii)} \quad \text{for any polynomial } P \text{ with } \lceil P \rceil < \lceil P_{i+1} \rceil \text{ we have } |P(\xi)| \geq c_3^{-1}|P_i(\xi)|.
\end{aligned}$$

Fix some $h \geq 1$. Consider the set of polynomials P with $\lceil P \rceil \leq h$. Their values at ξ are distinct. Hence there is a unique (up to the sign) polynomial P_1 with $\lceil P_1 \rceil \leq h$ and minimal absolute value at ξ. Now increase h until a polynomial P_2 with $\lceil P_2 \rceil \leq h$, $|P_2(\xi)| < c_3^{-1}|P_1(\xi)|$ appears. If there are several polynomials of this kind, pick one with minimal absolute value at ξ, etc. Repeating this process, we obtain the sequence of polynomials (3.1).

For any integer $i > 1$ we set $Q_i^{(0)}(x) = P_i(x)$. Write

$$Q_i^{(0)}(x) = b_n^{(0)}x^n + \ldots + b_1^{(0)}x + b_0^{(0)}.$$

By Lemma 2.1 there is an index $j_1 \in \{1, \ldots, n\}$ such that $|b_{j_1}^{(0)}| = \lceil Q_i^{(0)} \rceil$.

We now successively construct polynomials $Q_i^{(0)}, \dots, Q_i^{(n-1)}$ and distinct integers $j_1, \dots, j_n$ from $\{1, \dots, n\}$. Write

$$Q_i^{(\ell)}(x) = b_n^{(\ell)} x^n + \dots + b_1^{(\ell)} x + b_0^{(\ell)} \quad (\ell = 0, \dots, n-1).$$

The polynomials $Q_i^{(\ell)}$ and the integers $j_{\ell+1}$ (which we call the *indices of the* $\boldsymbol{Q_i}$*-system*) will have the following properties:

$$\begin{aligned} &\text{(i)} \quad |Q_i^{(\ell)}(\xi)| < c_3^{-1}\, |P_{i-1}(\xi)|, \\ &\text{(ii)} \quad |b_{j_\mu}^{(\ell)}| \le c_3^{-1}\, \boxed{Q_i^{(\mu-1)}} \quad (\mu = 1, \dots, \ell), \\ &\text{(iii)} \quad |b_{j_{\ell+1}}^{(\ell)}| = \boxed{Q_i^{(\ell)}}, \\ &\text{(iv)} \quad \boxed{Q_i^{(\ell)}} \le c_3^{\frac{\ell+1}{n-\ell}} \left(|P_{i-1}(\xi)| \prod_{\nu=0}^{\ell-1} \boxed{Q_i^{(\nu)}} \right)^{-\frac{1}{n-\ell}}, \end{aligned} \tag{3.2}$$

(if $\ell = 0$, we have (3.2)(i) and (3.2)(iii) only). Moreover, if for some ℓ with $0 \le \ell \le n-1$ there is a polynomial $Q(x) = b_n x^n + \dots + b_1 x + b_0$ satisfying

$$\begin{aligned} &|Q(\xi)| < c_3^{-1}\, |P_{i-1}(\xi)|, \\ &|b_{j_\mu}| \le c_3^{-1}\, \boxed{Q_i^{(\mu-1)}} \quad (\mu = 1, \dots, \ell), \end{aligned}$$

(if $\ell = 0$, we have $|Q(\xi)| < c_3^{-1} |P_{i-1}(\xi)|$ only), then $\boxed{Q} \ge \boxed{Q_i^{(\ell)}}$. In other words, $Q_i^{(\ell)}$ has minimum height among polynomials with (3.2)(i) and (3.2)(ii). We call this *the minimality property of* $\boldsymbol{Q_i^{(\ell)}}$.

Lemma 3.1. *For any integer $i > 1$ the polynomials $P_{i-1}, Q_i^{(0)}, \ldots, Q_i^{(n-2)}$ are linearly independent.*

Proof. Put

$$D = \begin{vmatrix} P_{i-1}(\xi) & Q_i^{(0)}(\xi) & \ldots & Q_i^{(n-2)}(\xi) \\ a_{j_1} & b_{j_1}^{(0)} & \ldots & b_{j_1}^{(n-2)} \\ \vdots & \vdots & & \vdots \\ a_{j_{n-1}} & b_{j_{n-1}}^{(0)} & \ldots & b_{j_{n-1}}^{(n-2)} \end{vmatrix}, \qquad d = \begin{vmatrix} b_{j_1}^{(0)} & \ldots & b_{j_1}^{(n-2)} \\ \vdots & & \vdots \\ b_{j_{n-1}}^{(0)} & \ldots & b_{j_{n-1}}^{(n-2)} \end{vmatrix},$$

where $a_{j_1}, \ldots, a_{j_{n-1}}$ are the coefficients of P_{i-1}. Let μ and ℓ be some integers with $1 \leq \mu \leq \ell \leq n-2$. By (3.2)(ii), (3.6) and the definition of c_3 we have

$$|b_{j_\mu}^{(\ell)}| \leq c_3^{-1} \boxed{Q_i^{(\mu-1)}} \leq c_3^{-1} \boxed{Q_i^{(\ell)}} \leq \frac{1}{n!} \boxed{Q_i^{(\ell)}},$$

hence

$$|d| > \prod_{\nu=0}^{n-2} |b_{j_{\nu+1}}^{(\nu)}| - \frac{1}{n} \prod_{\nu=0}^{n-2} \boxed{Q_i^{(\nu)}} = \frac{n-1}{n} \prod_{\nu=0}^{n-2} \boxed{Q_i^{(\nu)}} \tag{3.7}$$

by (3.2)(iii).

On the other hand, by (3.6) the absolute values of other minors from the last $n-1$ rows of D are $< (n-1)!\prod_{\nu=0}^{n-2}\overline{\left|Q_i^{(\nu)}\right|}$. By this, (3.2)(i), (3.7) and the definition of c_3 we get

$$\begin{aligned}|D| &> \frac{n-1}{n}|P_{i-1}(\xi)|\prod_{\nu=0}^{n-2}\overline{\left|Q_i^{(\nu)}\right|} - (n-1)!\left(\sum_{\nu=0}^{n-2}|Q_i^{(\nu)}(\xi)|\right)\prod_{\nu=0}^{n-2}\overline{\left|Q_i^{(\nu)}\right|}\\ &> \frac{n-1}{n}|P_{i-1}(\xi)|\prod_{\nu=0}^{n-2}\overline{\left|Q_i^{(\nu)}\right|} - \frac{n-1}{n}|P_{i-1}(\xi)|\prod_{\nu=0}^{n-2}\overline{\left|Q_i^{(\nu)}\right|}\\ &= 0.\end{aligned}$$

So, $D \neq 0$, therefore the polynomials $P_{i-1}, Q_i^{(0)}, \ldots, Q_i^{(n-2)}$ are linearly independent. □

4. Construction of polynomials $L_{i,\tau}$

Let i and τ be integers greater than 1 and let $\kappa_1, \dots, \kappa_{n-1}$ be the indices of the Q_τ-system. Consider the following system of n inequalities:

$$\begin{aligned}
&\left| P_{i-1}(\xi)x_1 + \sum_{\nu=0}^{n-2} Q_i^{(\nu)}(\xi)x_{\nu+2} \right| \le (2n)^{n/2} c_3^{n-1} |P_{i-1}(\xi)| \prod_{\nu=0}^{n-2} \overline{|Q_i^{(\nu)}|} \prod_{\nu=0}^{n-2} \overline{|Q_\tau^{(\nu)}|}^{-1}, \\
&\left| a_{\kappa_1}x_1 + \sum_{\nu=0}^{n-2} b_{\kappa_1}^{(\nu)} x_{\nu+2} \right| \le c_3^{-1} \overline{|Q_\tau^{(0)}|}, \\
&\vdots \\
&\left| a_{\kappa_{n-1}}x_1 + \sum_{\nu=0}^{n-2} b_{\kappa_{n-1}}^{(\nu)} x_{\nu+2} \right| \le c_3^{-1} \overline{|Q_\tau^{(n-2)}|}.
\end{aligned} \tag{4.1}$$

Lemma 4.1. *The system* (4.1) *has a nonzero integer solution* $(\tilde{x}_1, \dots, \tilde{x}_n)$ *which satisfies the following conditions:*

$$|\tilde{x}_1| \le R\, \overline{|P_{i-1}|}^{-1}, \qquad |\tilde{x}_\nu| \le R\, \overline{|Q_i^{(\nu-2)}|}^{-1} \quad (\nu = 2, \dots, n), \tag{4.2}$$

where

$$R = \max\left\{ 2^{n/2} n^{(n-1)/2} c_3^{n-1} \prod_{\nu=0}^{n-2} \overline{|Q_i^{(\nu)}|} \prod_{\nu=0}^{n-2} \overline{|Q_\tau^{(\nu)}|}^{-1} \overline{|P_{i-1}|}, \quad n^{-1/2} c_3^{-1} \overline{|Q_\tau^{(n-2)}|} \right\}. \tag{4.3}$$

5. Upper bounds for $|P_{i-1}(\xi)|$, $\prod_{\nu=0}^{n-2}\boxed{Q_i^{(\nu)}}$ and $\boxed{Q_i^{(n-2)}}$

Lemma 5.1. *Let i be an integer > 1. Then*

$$|P_{i-1}(\xi)| < \left(\xi^{n-1}\, c_3^{1-n(A-1)}\, \delta\right)^{-\frac{1}{A-2}} \boxed{P_i}^{-\omega}, \tag{5.5}$$

where $\delta = \delta(P_i)$. Suppose i_0 is an integer > 1 such that $\boxed{P_{i_0}} > H_2$. Then for any $i \geq i_0$ we have

$$\begin{aligned} &\text{(i)} && |P_{i-1}(\xi)| < \boxed{P_i}^{-\omega}, \\ &\text{(ii)} && |P_{i-1}(\xi)| < \boxed{P_i}^{-n}, \\ &\text{(iii)} && \prod_{\nu=0}^{n-2}\boxed{Q_i^{(\nu)}} < (2n)^{-n/2}(c_3c_4)^{-n+1}|P_{i-1}(\xi)|^{-\frac{A-2}{A-1}}, \\ &\text{(iv)} && \boxed{Q_i^{(n-2)}} < |P_{i-1}(\xi)|^{-\frac{A-2}{A-1}}\boxed{P_i}^{-n+2}. \end{aligned} \tag{5.6}$$

Suppose i and τ are integers such that $i > \tau \geq i_0$ and (4.9) hold. Then

$$\boxed{Q_\tau^{(n-2)}} < |P_{i-1}(\xi)|^{-\frac{A-2}{A-1}}\boxed{P_{i-1}}^{-n+2}. \tag{5.7}$$

6. Upper bounds for $|L_{i,\tau}(\xi)|$ and $|L'_{i,\tau}(\xi)|$

Lemma 6.1. *Let i_0 be an integer as in Lemma 5.1. Suppose i and τ are integers such that $i > \tau \geq i_0$ and (4.9) hold. Then the following estimates are valid:*

$$\begin{aligned} &\text{(i)} && |L_{i,\tau}(\xi)| < |P_{i-1}(\xi)|^{\frac{1}{A-1}}\boxed{P_{i-1}}^{-n+1}, \\ &\text{(ii)} && |L'_{i,\tau}(\xi)| < |P_{i-1}(\xi)|^{3-A-\frac{A-2}{A-1}}\boxed{P_i}^{-(n-2)(A-1)}\boxed{P_{i-1}}^{-n+2}. \end{aligned} \tag{6.1}$$

7. Lower bounds for $|P_{i-1}(\xi)|$

Put

$$\Phi(x) = \Phi(x, n, A) = \max\left\{n,\ 2A + n + x - 3 - 2\omega,\ A + \frac{n + x - 3 - \omega}{2}\right\}.$$

Proposition 7.1. *Let i_0 be an integer as in Lemma 5.1. Suppose k_0 is an integer such that*

$$\overline{|P_{k_0}|} > (n+1)^{n-1} M^n \quad \text{with} \quad M = \max\left\{c_6,\ c_2^{1/2} c_6^{-2} c_8,\ e^n \overline{|P_{i_0}|}\right\}. \tag{7.1}$$

If P_{i-1} is irreducible and has degree n for some $i > k_0$, then

$$|P_{i-1}(\xi)|^{-1} < \overline{|P_{i-1}|}^{\Phi(n)}. \tag{7.2}$$

If P_{i-1} is irreducible and has degree $< n$ or is reducible for some $i > k_0$, then

$$|P_{i-1}(\xi)|^{-1} < \overline{|P_{i-1}|}^{\Phi(n-1)}. \tag{7.3}$$

Proposition 7.4. *Let k_0 be an integer as in Proposition 7.1. If P_{i-1} is irreducible and has degree n for some $i > k_0$, then*

$$|P_{i-1}(\xi)|^{-1} < \overline{|P_i|}^{\frac{2A+n-2}{3}} \overline{|P_{i-1}|}^{\frac{n-1}{3}}. \tag{7.14}$$

Corollary 7.5. *Let k_0 be an integer as in Proposition 7.1. If P_{i-1} is irreducible and has degree n for some $i > k_0$, then for any ϱ with $0 \le \varrho \le 1$ we have*

$$|P_{i-1}(\xi)|^{-1} < \overline{|P_i|}^{\frac{2A+n-2}{3}(1-\varrho)} \overline{|P_{i-1}|}^{\frac{n-1}{3}(1-\varrho)+\Phi(n)\varrho}. \tag{7.15}$$

8. Proof of (4.10)

Lemma 8.1. *Let k_0 be an integer as in Proposition* **7.1.** *Suppose i and τ are integers such that $i > \tau \geq k_0$ and* **(4.9)** *hold. Then*

$$|L_{i,\tau}(\xi)| < |P_{i-1}(\xi)|^{A^2-3A+1}\boxed{P_i}^{(n-2)(A-1)^2}\boxed{P_{i-1}}^{(n-2)(A-1)}. \tag{8.1}$$

Proof. From (3.1)(ii) and (6.1)(i) it follows that for any α_1 and any nonnegative α_2 we have

$$|L_{i,\tau}(\xi)| < |P_{i-1}(\xi)|^{\frac{1}{A-1}+\alpha_1}|P_{i-1}(\xi)|^{-\alpha_1}\boxed{P_i}^{\alpha_2}\boxed{P_{i-1}}^{-n+1-\alpha_2}. \tag{8.2}$$

Put

$$\alpha_1 = -\frac{1}{A-1} + A^2 - 3A + 1. \tag{8.3}$$

Since $A > 3$ by (10.1)(iii), it follows that $\alpha_1 > 0$. We now distinguish two cases:

Case A: Suppose P_{i-1} is irreducible and has degree n. By (7.15) and (8.2) we have

$$|L_{i,\tau}(\xi)| < |P_{i-1}(\xi)|^{\frac{1}{A-1}+\alpha_1} \lceil P_i \rceil^{\frac{2A+n-2}{3}(1-\varrho)\alpha_1+\alpha_2} \lceil P_{i-1} \rceil^{\left(\frac{n-1}{3}(1-\varrho)+\Phi(n)\varrho\right)\alpha_1-n+1-\alpha_2}. \quad (8.4)$$

If $n = 3, 4, 5$, put

$$\varrho = 1 - \frac{3(n-2)(A-1)^2}{(2A+n-2)\alpha_1} \quad \text{and} \quad \alpha_2 = 0. \quad (8.5)$$

A straightforward calculation shows that $0 < \varrho < 1$. It follows from (8.3), (8.5) and (10.3)(iii) that

$$\begin{aligned} &\left(\frac{n-1}{3}(1-\varrho) + \Phi(n)\varrho\right)\alpha_1 - n + 1 \\ &= \frac{AT(A)}{(A-1)(A-2)(2A+n-2)} + (n-2)(A-1), \end{aligned}$$

which is $(n-2)(A-1)$ by (1.4). This and (8.3) – (8.5) give (8.1). Similarly, if $n > 5$, put

$$\varrho = 0 \quad \text{and} \quad \alpha_2 = \frac{n-1}{3}\alpha_1 - n + 1 - (n-2)(A-1). \quad (8.6)$$

One can show (see Lemma 10.3) that

$$\alpha_2 > 0 \quad \text{and} \quad \frac{2A+n-2}{3}\alpha_1 + \alpha_2 < (n-2)(A-1)^2. \quad (8.7)$$

From (8.3), (8.4), (8.6) and (8.7) follows (8.1).

Case B: Suppose P_{i-1} is irreducible and has degree $< n$ or is reducible. By (7.3) and (8.2) we have

$$|L_{i,\tau}(\xi)| < |P_{i-1}(\xi)|^{\frac{1}{A-1}+\alpha_1} \overline{|P_i|}^{\alpha_2} \overline{|P_{i-1}|}^{\Phi(n-1)\alpha_1 - n+1-\alpha_2}. \tag{8.8}$$

Put

$$\alpha_2 = (n-2)(A-1)^2. \tag{8.9}$$

A straightforward calculation shows that

$$\Phi(n-1)\alpha_1 - n + 1 - \alpha_2 < (n-2)(A-1)$$

if $n = 3, 4, 5$. Suppose $n > 5$. From (8.3), (8.9) and (10.3)(ii) it follows that

$$\Phi(n-1)\alpha_1 - n + 1 - \alpha_2 = \frac{T(A)}{(A-1)(A-2)} + (n-2)(A-1),$$

which is $(n-2)(A-1)$ by (1.4). This, (8.3), (8.8) and (8.9) give (8.1). □

Corollary 8.2. *Let i and τ be integers as in Lemma* 8.1. *Then $L_{i,\tau}$ satisfies* (4.10).

Proof. If we raise both sides of (6.1)(ii) to the power $-A+1$, we obtain

$$|L'_{i,\tau}(\xi)|^{-A+1} > |P_{i-1}(\xi)|^{A^2-3A+1} \overline{|P_i|}^{(n-2)(A-1)^2} \overline{|P_{i-1}|}^{(n-2)(A-1)}.$$

Combining this with (8.1), we get (4.10). □

9. Proof of the Theorem

Lemma 9.1. *Let* i *and* τ *be integers as in Lemma* **8.1.** *Then*

$$|P_{\tau-1}(\xi)|\prod_{\nu=0}^{n-2}\overline{\left|Q_\tau^{(\nu)}\right|} \le (2n)^{n/2}c_3^n|P_{i-1}(\xi)|\prod_{\nu=0}^{n-2}\overline{\left|Q_i^{(\nu)}\right|}.$$

Proof. Suppose to the contrary

$$|P_{\tau-1}(\xi)|\prod_{\nu=0}^{n-2}\overline{\left|Q_\tau^{(\nu)}\right|} > (2n)^{n/2}c_3^n|P_{i-1}(\xi)|\prod_{\nu=0}^{n-2}\overline{\left|Q_i^{(\nu)}\right|}. \tag{9.1}$$

By (4.1) and (4.8) we have

$$|L_{i,\tau}(\xi)| \le (2n)^{n/2}c_3^{n-1}|P_{i-1}(\xi)|\prod_{\nu=0}^{n-2}\overline{\left|Q_i^{(\nu)}\right|}\prod_{\nu=0}^{n-2}\overline{\left|Q_\tau^{(\nu)}\right|}^{-1},$$

which is $< c_3^{-1}|P_{\tau-1}(\xi)|$ by (9.1). This, (4.1) and (4.8) imply

$$\begin{aligned} &\text{(i)} \quad |L_{i,\tau}(\xi)| < c_3^{-1}|P_{\tau-1}(\xi)|, \\ &\text{(ii)} \quad |d_{\kappa_\nu}| \le c_3^{-1}\overline{\left|Q_\tau^{(\nu-1)}\right|} \quad (\nu = 1,\dots,n-1), \end{aligned} \tag{9.2}$$

where $d_{\kappa_1},\dots,d_{\kappa_{n-1}}$ are coefficients of $L_{i,\tau}$.

From (9.2) by the minimality property of $Q_\tau^{(n-2)}$ we get

$$\lceil L_{i,\tau} \rceil \geq \lceil Q_\tau^{(n-2)} \rceil. \tag{9.3}$$

This, (3.6), (9.2)(ii) and the definition of c_3 give

$$|d_{\kappa\nu}| \leq c_3^{-1} \lceil Q_\tau^{(\nu-1)} \rceil \leq c_3^{-1} \lceil Q_\tau^{(n-2)} \rceil \leq c_3^{-1} \lceil L_{i,\tau} \rceil \leq \xi^{n-1} \lceil L_{i,\tau} \rceil \quad (\nu = 1, \ldots, n-1).$$

Therefore $L_{i,\tau}$ satisfies the conditions of Lemma 2.2, by which

$$|L'_{i,\tau}(\xi)| > \xi^{n-1} \lceil L_{i,\tau} \rceil. \tag{9.4}$$

We also note that by (3.6) and (9.3) we have $\lceil L_{i,\tau} \rceil \geq \lceil P_\tau \rceil$, which is $> H_2$, since $\tau \geq k_0$. Therefore we can apply (5.3) to $L_{i,\tau}$. From this and (9.4) it follows that

$$|L_{i,\tau}(\xi)| > c_8 |L'_{i,\tau}(\xi)| \lceil L_{i,\tau} \rceil^{-A} > c_8 \xi^{(n-1)A} |L'_{i,\tau}(\xi)|^{-A+1},$$

which is $> |L'_{i,\tau}(\xi)|^{-A+1}$ by (5.2). We obtain a contradiction with (4.10). $\square$

Proof of the Theorem. Choose an increasing sequence of integers $\{m_t\}$ such that $k_0 = m_1 < m_2 < \ldots$ and

$$\lceil P_{m_{t+1}-1} \rceil \leq c_4 \lceil P_{m_t} \rceil < \lceil P_{m_{t+1}} \rceil, \quad t = 1, 2, \ldots \tag{9.5}$$

By Lemma 9.1 we have

$$|P_{m_t-1}(\xi)| \prod_{\nu=0}^{n-2} \lceil Q_{m_t}^{(\nu)} \rceil \leq (2n)^{n/2} c_3^n |P_{m_{t+1}-1}(\xi)| \prod_{\nu=0}^{n-2} \lceil Q_{m_{t+1}}^{(\nu)} \rceil, \quad t = 1, 2, \ldots$$

Let ℓ be some integer ≥ 1. If we multiply these inequalities together for all t with $1 \leq t \leq \ell$, we obtain

$$|P_{m_1-1}(\xi)| \prod_{\nu=0}^{n-2} \lceil Q_{m_1}^{(\nu)} \rceil \leq (2n)^{n\ell/2} c_3^{n\ell} |P_{m_{\ell+1}-1}(\xi)| \prod_{\nu=0}^{n-2} \lceil Q_{m_{\ell+1}}^{(\nu)} \rceil,$$

hence

$$|P_{m_1-1}(\xi)| < (2n)^{n\ell/2} c_3^{n\ell} |P_{m_{\ell+1}-1}(\xi)| \prod_{\nu=0}^{n-2} \lceil Q_{m_{\ell+1}}^{(\nu)} \rceil. \tag{9.6}$$

Substituting (5.6)(iii) into (9.6), using the definitions of c_3, c_4 and then applying (5.6)(i), we get

$$|P_{m_1-1}(\xi)| < (2n)^{n\ell/2} c_3^{n\ell} |P_{m_{\ell+1}-1}(\xi)|^{\frac{1}{A-1}} < (2n)^{n\ell/2} c_3^{n\ell} \lceil P_{m_{\ell+1}} \rceil^{-\frac{\omega}{A-1}}. \tag{9.7}$$

By the right-hand side of (9.5) we have

$$\lceil P_{m_{t+1}} \rceil^{-1} < c_4^{-1} \lceil P_{m_t} \rceil^{-1}, \quad t = 1, 2, \ldots$$

If we multiply these inequalities together for all t with $1 \le t \le \ell$, we obtain

$$\lceil P_{m_{\ell+1}} \rceil^{-1} < c_4^{-\ell} \lceil P_{m_1} \rceil^{-1} < c_4^{-\ell}.$$

Using this in (9.7) and keeping in mind the definitions of c_4, ω, we get

$$|P_{m_1-1}(\xi)| < (2n)^{n\ell/2}\, c_3^{n\ell}\, c_4^{-\frac{\omega}{A-1}\ell} = 2^{-\ell}.$$

Letting $\ell \to \infty$, we come to a contradiction. Thus, the assumption (1.5) can not be true. This completes the proof of the theorem. $\square$

n	Th. 3, 1961	Th. 4, 1961	Th. 5, 1993	Th. 6, 2007	Conjecture
3	3	3.28	3.5	3.73	4
4	3.5	3.82	4.12	4.45	5
5	4	4.35	4.71	5.14	6
6	4.5	4.87	5.28	5.76	7
7	5	5.39	5.84	6.36	8
8	5.5	5.9	6.39	6.93	9
9	6	6.41	6.93	7.50	10
10	6.5	6.92	7.47	8.06	11
15	9	9.44	10.09	10.77	16
20	11.5	11.95	12.67	13.40	21
50	26.5	26.98	27.84	28.70	51
100	51.5	51.99	52.92	53.84	101

<u>THEOREM 7</u> (Davenport - Schmidt, 1968): Let ξ be real, but not algebraic of degree $\leq$ 2. Then there are infinitely many algebraic integers α of degree ≤ 3 which satisfy

$$0 < |\xi - \alpha| \ll H(\alpha)^{-2.618\ldots}$$

Theorem 7 (Davenport - Schmidt, 1968): Let $n \geq 3$. Let ξ be real, but not algebraic of degree ≤ 2 if $n = 3, 4$. Let also ξ be real, but not algebraic of degree $\leq (n-1)/2$ if $n \geq 5$. Then there are infinitely many algebraic integers α of degree $\leq n$ which satisfy

$$|\xi - \alpha| \ll H(\alpha)^{-A(n)},$$

where

$$A(n) = \begin{cases} \dfrac{3+\sqrt{5}}{2} & \text{if } n = 3 \\ 4 & \text{if } n = 4 \\ \left\lfloor \dfrac{n+1}{2} \right\rfloor & \text{if } n > 4. \end{cases}$$

THEOREM 7 (Davenport - Schmidt, 1968): Let ξ be real, but not algebraic of degree ≤ 2. Then there are infinitely many algebraic integers α of degree ≤ 3 which satisfy

$$0 < |\xi - \alpha| \ll H(\alpha)^{-2.618\ldots}$$

CONJECTURE: Let ξ be real, but is not algebraic of degree ≤ 2. Suppose $\epsilon > 0$. Then there are infinitely many real algebraic integers α of degree ≤ 3 with

$$|\xi - \alpha| \ll H(\alpha)^{-3+\epsilon}$$

THEOREM 8 (Roy, 2001): There exist real numbers ξ such that for any algebraic integer α of degree ≤ 3, we have

$$|\xi - \alpha| \gg H(\alpha)^{-2.618\ldots}$$

<u>Theorem 8</u> **(Roy, 2001): There exist real numbers ξ such that for any algebraic integer α of degree ≤ 3, we have**

$$|\xi - \alpha| \gg H(\alpha)^{-2.618\ldots}$$

$$\xi = \cfrac{1}{3 + \cfrac{1}{2 + \cfrac{1}{4 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{4 + \cfrac{1}{2 + \cfrac{1}{4 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{4+}}}}}}}}}}}$$

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Consider a sequence of polynomials $P_i \in \mathbb{Z}[x]$ of degree $\leq n$ such that

(i) $\frac{1}{2} > |P_1(\xi)| > |P_2(\xi)| > \ldots > |P_i(\xi)| > \ldots,$

(ii) $\lceil P_1 \rceil < \lceil P_2 \rceil < \ldots < \lceil P_i \rceil < \ldots,$

(iii) **for any P with $\lceil P \rceil < \lceil P_{i+1} \rceil$ we have $|P(\xi)| \geq |P_i(\xi)|$.**

<u>EXAMPLE</u>: **Let $n = 1$ and $\xi = \dfrac{1+\sqrt{5}}{2}$. Then**

$P_1(x) = x - 2$	
$P_2(x) = 2x - 3$	
$P_3(x) = 3x - 5$	$P_3 = P_2 + P_1$
$P_4(x) = 5x - 8$	$P_4 = P_3 + P_2$
$P_5(x) = 8x - 13$	$P_5 = P_4 + P_3$
$P_6(x) = 13x - 21$	$P_6 = P_5 + P_4$
$P_7(x) = 21x - 34$	$P_7 = P_6 + P_5$

Consider a sequence of polynomials $P_i \in \mathbb{Z}[x]$ of degree $\leq n$ such that

(i) $\frac{1}{2} > |P_1(\xi)| > |P_2(\xi)| > \ldots > |P_i(\xi)| > \ldots,$

(ii) $\overline{|P_1|} < \overline{|P_2|} < \ldots < \overline{|P_i|} < \ldots,$

(iii) **for any P with $\overline{|P|} < \overline{|P_{i+1}|}$ we have $|P(\xi)| \geq |P_i(\xi)|$.**

<u>Example</u>: **Let $n = 1$ and $\xi = 1 + \sqrt{2}$. Then**

$P_1(x) = 2x - 5$	
$P_2(x) = 5x - 12$	
$P_3(x) = 12x - 29$	$P_3 = 2P_2 + P_1$
$P_4(x) = 29x - 70$	$P_4 = 2P_3 + P_2$
$P_5(x) = 70x - 169$	$P_5 = 2P_4 + P_3$
$P_6(x) = 169x - 408$	$P_6 = 2P_5 + P_4$
$P_7(x) = 408x - 985$	$P_7 = 2P_6 + P_5$

Consider a sequence of polynomials $P_i \in \mathbb{Z}[x]$ of degree $\leq n$ such that

(i) $\frac{1}{2} > |P_1(\xi)| > |P_2(\xi)| > \ldots > |P_i(\xi)| > \ldots,$

(ii) $\lceil P_1 \rceil < \lceil P_2 \rceil < \ldots < \lceil P_i \rceil < \ldots,$

(iii) **for any** P **with** $\lceil P \rceil < \lceil P_{i+1} \rceil$ **we have** $|P(\xi)| \geq |P_i(\xi)|$.

Example: **Let $n = 1$ and $\xi = 1 + \sqrt{3}$. Then**

$P_1(x) = x - 3$	
$P_2(x) = 3x - 8$	
$P_3(x) = 4x - 11$	$P_3 = P_2 + P_1$
$P_4(x) = 11x - 30$	$P_4 = 2P_3 + P_2$
$P_5(x) = 15x - 41$	$P_5 = P_4 + P_3$
$P_6(x) = 41x - 112$	$P_6 = 2P_5 + P_4$
$P_7(x) = 56x - 153$	$P_7 = P_6 + P_5$

Consider a sequence of polynomials $P_i \in \mathbb{Z}[x]$ of degree $\leq n$ such that

(i) $\frac{1}{2} > |P_1(\xi)| > |P_2(\xi)| > \ldots > |P_i(\xi)| > \ldots,$

(ii) $\lceil P_1 \rceil < \lceil P_2 \rceil < \ldots < \lceil P_i \rceil < \ldots,$

(iii) for any P with $\lceil P \rceil < \lceil P_{i+1} \rceil$ we have $|P(\xi)| \geq |P_i(\xi)|$.

<u>Example</u>: Let $n = 1$ and let ξ be an "extreme" number. Then

$P_1(x) = 3x - 1$	
$P_2(x) = 7x - 2$	
$P_3(x) = 31x - 9$	$P_3 = 4P_2 + P_1$
$P_4(x) = 69x - 20$	$P_4 = 2P_3 + P_2$
$P_5(x) = 169x - 49$	$P_5 = 2P_4 + P_3$
$P_6(x) = 745x - 216$	$P_6 = 4P_5 + P_4$
$P_7(x) = 1659x - 481$	$P_7 = 2P_6 + P_5$

Consider a sequence of polynomials $P_i \in \mathbb{Z}[x]$ of degree $\leq n$ such that

(i) $\frac{1}{2} > |P_1(\xi)| > |P_2(\xi)| > \ldots > |P_i(\xi)| > \ldots,$

(ii) $\lceil P_1 \rceil < \lceil P_2 \rceil < \ldots < \lceil P_i \rceil < \ldots,$

(iii) **for any** P **with** $\lceil P \rceil < \lceil P_{i+1} \rceil$ **we have** $|P(\xi)| \geq |P_i(\xi)|$.

<u>EXAMPLE</u>: **Let $n = 1$ and $\xi \in R$. Then**

$$\xi = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cfrac{1}{a_5 + \cfrac{1}{a_6 + \cfrac{1}{a_7 + \ldots}}}}}}}$$

$$P_1(x) = p_1 x - q_1$$

$$P_2(x) = p_2 x - q_2$$

$$P_3(x) = p_3 x - q_3 \qquad P_3 = a_3 P_2 + P_1$$

$$P_4(x) = p_4 x - q_4 \qquad P_4 = a_4 P_3 + P_2$$

$$P_5(x) = p_5 x - q_5 \qquad P_5 = a_5 P_4 + P_3$$

$$P_6(x) = p_6 x - q_6 \qquad P_6 = a_6 P_5 + P_4$$

$$P_7(x) = p_7 x - q_7 \qquad P_7 = a_7 P_6 + P_5$$

Let $\xi = 1.2599...$ be a root of $f(x) = x^3 - 2$. Then

1	$-x+1$	$[1,-1,0]$	0.51984
2	$-x^2+2\,x-1$	$[1,-1,-1]$	0.40535
3	$-3\,x^2+3\,x+1$	$[1,5,4]$	0.33364
4	$-6\,x^2+2\,x+7$	$[1,-3,3]$	0.40621
5	$-13\,x^2+14\,x+3$	$[1,4,0]$	1.0027
6	$-8\,x^2-5\,x+19$	$[1,3,-1]$	0.53385
7	$-27\,x^2+11\,x+29$	$[1,1,1]$	1.1784
8	$3\,x^2+24\,x-35$	$[1,-2,0]$	0.55812
9	$-21\,x^2+59\,x-41$	$[0,1,2]$	0.44907
10	$-60\,x^2+201\,x-158$	$[5,13,4]$	4.6835
11	$81\,x^2-260\,x+199$	$[5,14,-6]$	1.3921
12	$465\,x^2-1501\,x+1153$	$[5,4,10]$	25.553
13	$-546\,x^2+1761\,x-1352$	$[5,-28,2]$	28.800
14	$1011\,x^2-3262\,x+2505$	$[5,6,4]$	21.834
15	$4590\,x^2-14809\,x+11372$	$[5,-8,7]$	236.92

Let $\xi = 1.2599...$ be a root of $f(x) = x^3 - 2$. Then

1	$-x+1$	$[1,-1,0]$	0.51984
2	$-x^2+2\,x-1$	$[1,-1,-1]$	0.40535
3	$-3\,x^2+3\,x+1$	$[1,5,4]$	0.33364
4	$-6\,x^2+2\,x+7$	$[1,-3,3]$	0.40621
5	$-13\,x^2+14\,x+3$	$[1,4,0]$	1.0027
6	$-8\,x^2-5\,x+19$	$[1,3,-1]$	0.53385
7	$-27\,x^2+11\,x+29$	$[1,1,1]$	1.1784
8	$3\,x^2+24\,x-35$	$[1,-2,0]$	0.55812
9	$-21\,x^2+59\,x-41$	$[0,1,2]$	0.44907
10	$-80\,x^2+100\,x+1$	$[1,3,3]$	0.34167
11	$180\,x^2-99\,x-161$	$[1,3,-3]$	0.36886
12	$-341\,x^2+459\,x-37$	$[1,4,0]$	1.0443
13	$-279\,x^2-62\,x+521$	$[1,-3,-1]$	0.49699
14	$800\,x^2-496\,x-645$	$[1,-1,-1]$	1.0764
15	$217\,x^2+583\,x-1079$	$[1,-2,0]$	0.56745
16	$-366\,x^2+1662\,x-1513$	$[0,-1,2]$	0.49303
17	$-2028\,x^2+3175\,x-781$	$[1,-3,3]$	0.36585
18	$5203\,x^2-3956\,x-3275$	$[1,3,-3]$	0.34332
19	$9159\,x^2-681\,x-13681$	$[1,3,3]$	0.45334
20	$-22840\,x^2+18999\,x+12319$	$[1,-4,0]$	1.0145

Let $\xi = 0.5436...$ be a root of $f(x) = x^3 + x^2 + x - 1$. Then

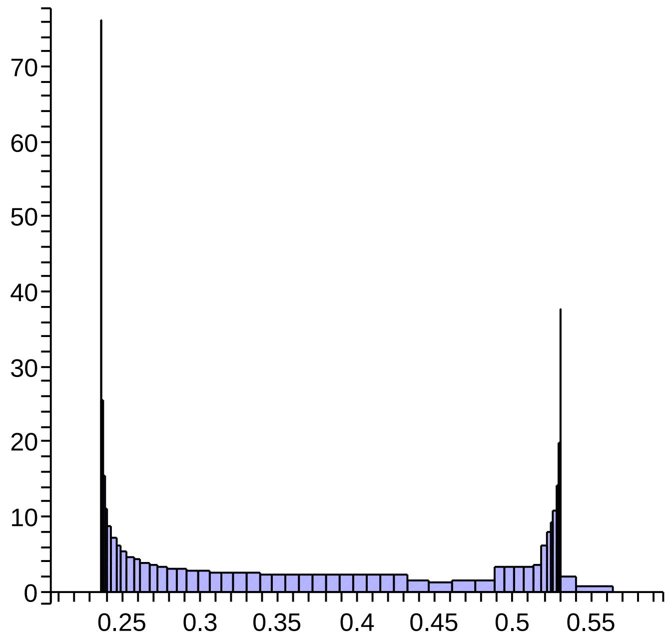
1	$-x^2 - x + 1$	$[1, -1, -1]$	0.48214
2	$-2x^2 + x$	$[1, -2, 0]$	0.23753
3	$3x^2 + 2x - 2$	$[1, -2, 0]$	0.43909
4	$-4x^2 + 4x - 1$	$[1, 3, 1]$	0.25195
5	$-8x^2 - 3x + 4$	$[1, -2, 0]$	0.36944
6	$-5x^2 - 12x + 8$	$[1, -2, 2]$	0.52585
7	$-7x^2 + 13x - 5$	$[1, 1, -1]$	0.29817
8	$-20x^2 - 2x + 7$	$[1, 1, 1]$	0.30221
9	$-18x^2 - 27x + 20$	$[1, -1, 1]$	0.52701
10	$-9x^2 + 38x - 18$	$[1, 1, -1]$	0.36462
11	$-47x^2 + 9x + 9$	$[1, 1, 1]$	0.25421
12	$-56x^2 - 56x + 47$	$[1, -1, 1]$	0.49437
13	$-103x^2 + 56x$	$[1, 2, 0]$	0.23684
14	$-159x^2 - 103x + 103$	$[1, 2, 0]$	0.43562
15	$-206x^2 + 215x - 56$	$[1, 3, -1]$	0.25421
16	$-421x^2 - 150x + 206$	$[1, 2, 0]$	0.36462
17	$271x^2 + 627x - 421$	$[1, 2, -2]$	0.52701
18	$-356x^2 + 692x - 271$	$[1, 1, 1]$	0.30221
19	$-1048x^2 - 85x + 356$	$[1, -1, 1]$	0.29817
20	$963x^2 + 1404x - 1048$	$[1, 1, -1]$	0.52582

Let $\xi = 0.6180...$ be a root of $f(x) = x^2 + x - 1$. Then

-1	-1		1
0	x		0.61803
1	$x - 1$	$[1, 1, 1]$	0.54018
2	$2\,x - 1$	$[1, 1, 1]$	0.47214
3	$3\,x - 2$	$[1, 1, 1]$	0.43769
4	$5\,x - 3$	$[1, 1, 1]$	0.45085
5	$8\,x - 5$	$[1, 1, 1]$	0.44582
6	$13\,x - 8$	$[1, 1, 1]$	0.44774
7	$21\,x - 13$	$[1, 1, 1]$	0.44701
8	$34\,x - 21$	$[1, 1, 1]$	0.44729
9	$55\,x - 34$	$[1, 1, 1]$	0.44718
10	$89\,x - 55$	$[1, 1, 1]$	0.44722
11	$144\,x - 89$	$[1, 1, 1]$	0.44721
12	$233\,x - 144$	$[1, 1, 1]$	0.44722
13	$377\,x - 233$	$[1, 1, 1]$	0.44721
14	$610\,x - 377$	$[1, 1, 1]$	0.44721
15	$987\,x - 610$	$[1, 1, 1]$	0.44721
16	$1597\,x - 987$	$[1, 1, 1]$	0.44721
17	$2584\,x - 1597$	$[1, 1, 1]$	0.44721
18	$4181\,x - 2584$	$[1, 1, 1]$	0.44721
19	$6765\,x - 4181$	$[1, 1, 1]$	0.44721
20	$10946\,x - 6765$	$[1, 1, 1]$	0.44721

Let $\xi = 0.5436...$ be a root of $f(x) = x^3 + x^2 + x - 1$. Then

1	$-x^2 - x + 1$	$[1, -1, -1]$	0.48214
2	$-2x^2 + x$	$[1, -2, 0]$	0.23753
3	$3x^2 + 2x - 2$	$[1, -2, 0]$	0.43909
4	$-4x^2 + 4x - 1$	$[1, 3, 1]$	0.25195
5	$-8x^2 - 3x + 4$	$[1, -2, 0]$	0.36944
6	$-5x^2 - 12x + 8$	$[1, -2, 2]$	0.52585
7	$-7x^2 + 13x - 5$	$[1, 1, -1]$	0.29817
8	$-20x^2 - 2x + 7$	$[1, 1, 1]$	0.30221
9	$-18x^2 - 27x + 20$	$[1, -1, 1]$	0.52701
10	$-9x^2 + 38x - 18$	$[1, 1, -1]$	0.36462
11	$-47x^2 + 9x + 9$	$[1, 1, 1]$	0.25421
12	$-56x^2 - 56x + 47$	$[1, -1, 1]$	0.49437
13	$-103x^2 + 56x$	$[1, 2, 0]$	0.23684
14	$-159x^2 - 103x + 103$	$[1, 2, 0]$	0.43562
15	$-206x^2 + 215x - 56$	$[1, 3, -1]$	0.25421
16	$-421x^2 - 150x + 206$	$[1, 2, 0]$	0.36462
17	$271x^2 + 627x - 421$	$[1, 2, -2]$	0.52701
18	$-356x^2 + 692x - 271$	$[1, 1, 1]$	0.30221
19	$-1048x^2 - 85x + 356$	$[1, -1, 1]$	0.29817
20	$963x^2 + 1404x - 1048$	$[1, 1, -1]$	0.52582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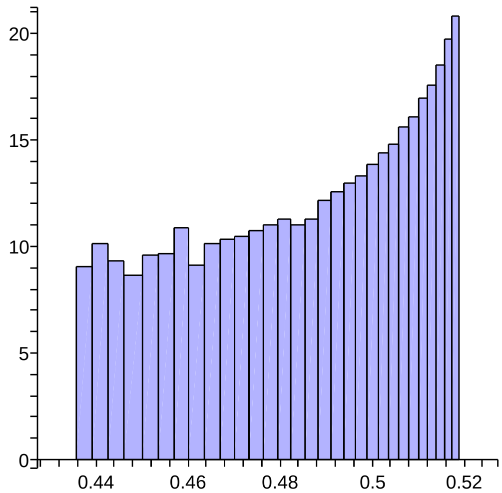

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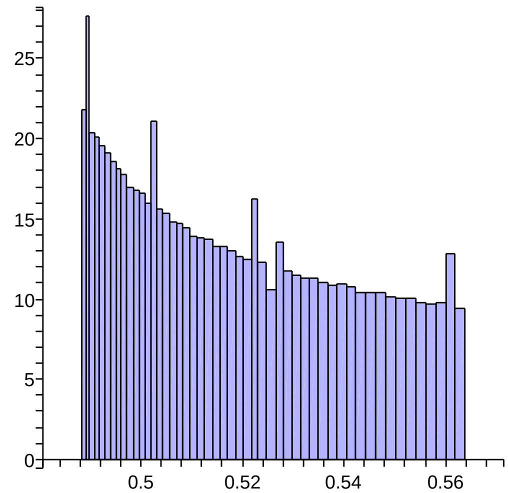

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0.54
0.56

Let $\xi = 0.6180...$ be a root of $f(x) = x^2 + x - 1$. Then

-1	-1			-1
0	x	$-x$		0.61803
1	$x-1$	$-x$	$[1,1,1]$	0.54018
2	$2x-1$	$-x$	$[1,1,1]$	0.47214
3	$3x-2$	$-x$	$[1,1,1]$	0.43769
4	$5x-3$	$-x$	$[1,1,1]$	0.45085
5	$8x-5$	$-x$	$[1,1,1]$	0.44582
6	$13x-8$	$-x$	$[1,1,1]$	0.44774
7	$21x-13$	$-x$	$[1,1,1]$	0.44701
8	$34x-21$	$-x$	$[1,1,1]$	0.44729
9	$55x-34$	$-x$	$[1,1,1]$	0.44718
10	$89x-55$	$-x$	$[1,1,1]$	0.44722
11	$144x-89$	$-x$	$[1,1,1]$	0.44721
12	$233x-144$	$-x$	$[1,1,1]$	0.44722
13	$377x-233$	$-x$	$[1,1,1]$	0.44721
14	$610x-377$	$-x$	$[1,1,1]$	0.44721
15	$987x-610$	$-x$	$[1,1,1]$	0.44721
16	$1597x-987$	$-x$	$[1,1,1]$	0.44721
17	$2584x-1597$	$-x$	$[1,1,1]$	0.44721
18	$4181x-2584$	$-x$	$[1,1,1]$	0.44721
19	$6765x-4181$	$-x$	$[1,1,1]$	0.44721
20	$10946x-6765$	$-x$	$[1,1,1]$	0.44721

Let $\xi = 0.5436...$ be a root of $f(x) = x^3 + x^2 + x - 1$. Then

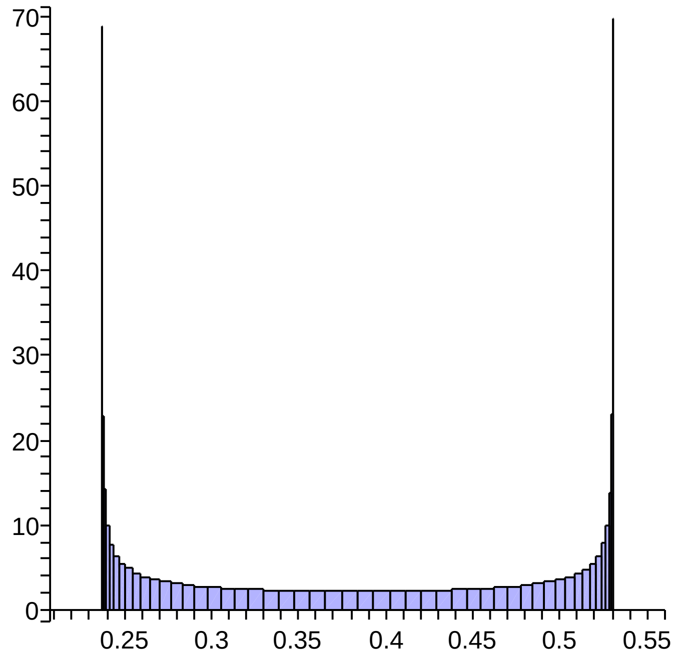
1	$-x^2 - x + 1$	$[1, -1, -1]$	x	0.48214
2	$-2\,x^2 + x$	$[1, -2, 0]$	$-x^2$	0.23753
3	$3\,x^2 + 2\,x - 2$	$[1, -2, 0]$	x	0.43909
4	$-4\,x^2 + 4\,x - 1$	$[1, 3, 1]$	x^2	0.25195
5	$-8\,x^2 - 3\,x + 4$	$[1, -2, 0]$	$-x$	0.36944
6	$-5\,x^2 - 12\,x + 8$	$[1, -2, 2]$	$-x$	0.52585
7	$-7\,x^2 + 13\,x - 5$	$[1, 1, -1]$	x	0.29817
8	$-20\,x^2 - 2\,x + 7$	$[1, 1, 1]$	$-x$	0.30221
9	$-18\,x^2 - 27\,x + 20$	$[1, -1, 1]$	$-x$	0.52701
10	$-9\,x^2 + 38\,x - 18$	$[1, 1, -1]$	x	0.36462
11	$-47\,x^2 + 9\,x + 9$	$[1, 1, 1]$	$-x$	0.25421
12	$-56\,x^2 - 56\,x + 47$	$[1, -1, 1]$	$-x$	0.49437
13	$-103\,x^2 + 56\,x$	$[1, 2, 0]$	$-x^2$	0.23684
14	$-159\,x^2 - 103\,x + 103$	$[1, 2, 0]$	$-x$	0.43562
15	$-206\,x^2 + 215\,x - 56$	$[1, 3, -1]$	$-x^2$	0.25421
16	$-421\,x^2 - 150\,x + 206$	$[1, 2, 0]$	$-x$	0.36462
17	$271\,x^2 + 627\,x - 421$	$[1, 2, -2]$	x	0.52701
18	$-356\,x^2 + 692\,x - 271$	$[1, 1, 1]$	$-x$	0.30221
19	$-1048\,x^2 - 85\,x + 356$	$[1, -1, 1]$	$-x$	0.29817
20	$963\,x^2 + 1404\,x - 1048$	$[1, 1, -1]$	x	0.52582

Let $\xi = 0.5436...$ be a root of $f(x) = x^3 + x^2 + x - 1$. Then

21	$-441\,x^2 + 2011\,x - 963$	$[1, 1, 1]$	$-x$	0.36950
22	$-2452\,x^2 + 522\,x + 441$	$[1, -1, 1]$	$-x$	0.25197
23	$-2893\,x^2 + 2533\,x - 522$	$[0, 1, 1]$	$-x^2 - x$	0.49147
24	$2974\,x^2 + 2893\,x - 2452$	$[1, 1, -1]$	$-1/2\,x^2 - 1/2$	0.49108
25	$-5345\,x^2 + 3055\,x - 81$	$[1, 1, 0]$	x^2	0.23692
26	$8400\,x^2 + 5264\,x - 5345$	$[1, 2, -1]$	x	0.43099
27	$10609\,x^2 - 11536\,x + 3136$	$[1, -3, -1]$	$-x^2$	0.25660
28	$-22145\,x^2 - 7473\,x + 10609$	$[1, -2, 0]$	x	0.35977
29	$-14672\,x^2 - 32754\,x + 22145$	$[1, 2, 2]$	$-x$	0.52805
30	$18082\,x^2 - 36817\,x + 14672$	$[1, -1, 1]$	$-x$	0.30634
31	$-54899\,x^2 - 3410\,x + 18082$	$[1, 1, -1]$	x	0.29423
32	$-51489\,x^2 - 72981\,x + 54899$	$[1, 1, 1]$	$-x$	0.52447
33	$21492\,x^2 - 106388\,x + 51489$	$[1, -1, 1]$	$-x$	0.37440
34	$-127880\,x^2 + 29997\,x + 21492$	$[1, 1, -1]$	x	0.24987
35	$-149372\,x^2 + 136385\,x - 29997$	$[0, -1, 1]$	$-x^2 - x$	0.49498
36	$-157877\,x^2 - 149372\,x + 127880$	$[1, 1, 1]$	$1/2\,x^2 + 1/2$	0.48766
37	$-277252\,x^2 + 166382\,x - 8505$	$[1, 1, 0]$	$-x^2$	0.23717
38	$443634\,x^2 + 268747\,x - 277252$	$[1, -2, -1]$	x	0.42632
39	$-545999\,x^2 + 618521\,x - 174887$	$[1, 3, 1]$	x^2	0.25913
40	$-1164520\,x^2 - 371112\,x + 545999$	$[1, -2, 0]$	$-x$	0.35494

Let $\xi = 0.5436...$ be a root of $f(x) = x^3 + x^2 + x - 1$. Then

1	x	$[1,-1,-1,1]$	x	0.54369
2	x^2	$[1,-1,-1,1]$	x	0.29560
3	$-x^2 - x + 1$	$[1,-1,-1,1]$	x	0.48214
4	$2x - 1$	$[1,-1,-1,1]$	x	0.43689
5	$2x^2 - x$	$[1,-1,-1,1]$	x	0.23753
6	$-3x^2 - 2x + 2$	$[1,-1,-1,1]$	x	0.43909
7	$x^2 + 5x - 3$	$[1,-1,-1,1]$	x	0.49150
8	$4x^2 - 4x + 1$	$[1,-1,-1,1]$	x	0.25195
9	$-8x^2 - 3x + 4$	$[1,-1,-1,1]$	x	0.36944
10	$5x^2 + 12x - 8$	$[1,-1,-1,1]$	x	0.52585
11	$7x^2 - 13x + 5$	$[1,-1,-1,1]$	x	0.29817
12	$-20x^2 - 2x + 7$	$[1,-1,-1,1]$	x	0.30221
13	$18x^2 + 27x - 20$	$[1,-1,-1,1]$	x	0.52701
14	$9x^2 - 38x + 18$	$[1,-1,-1,1]$	x	0.36462
15	$-47x^2 + 9x + 9$	$[1,-1,-1,1]$	x	0.25421
16	$56x^2 + 56x - 47$	$[1,-1,-1,1]$	x	0.49437
17	$-103x + 56$	$[1,-1,-1,1]$	x	0.43562
18	$-103x^2 + 56x$	$[1,-1,-1,1]$	x	0.23684
19	$159x^2 + 103x - 103$	$[1,-1,-1,1]$	x	0.43562
20	$-56x^2 - 262x + 159$	$[1,-1,-1,1]$	x	0.49437



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Consider a polynomial $f(x) = x^3 + a_2x^2 + a_1x - 1$ in $\mathbb{Z}[x]$ irreducible over $\mathbb{Q}$. Let ξ, ξ_1 and ξ_2 be its roots. Assume that ξ is real and ξ_1, ξ_2 are complex. We also assume that $|\xi| < 1$.

We introduce a sequence of polynomials $P_n(x) = a_n + b_nx + c_nx^2$ in $\mathbb{Z}[x]$, defined by $P_n(x) = x^n \mod f$. Put

$$K_n = |P_n(\xi)|(a_n^2 + b_n^2 + c_n^2)$$

and $I = \inf_n K_n, \quad S = \sup_n K_n, \quad \tilde{K}_n = (K_n - I)/S.$

<u>THEOREM (HENSLEY-T.)</u>: A frequency curve for $\{\tilde{K}_n\}$ is $f(x) = \dfrac{1}{\pi\sqrt{x(1-x)}}$. Moreover, $I = \inf_n K_n$ and $S = \sup_n K_n$ are roots of $p(x) = c_6x^6 + \dots c_2x^2 + c_1x + c_0$ with

$$\begin{aligned}
c_6 &= d^3\\
c_5 &= 4\left({a_2}^4 - {a_2}^3 - 4\,{a_2}^2a_1 + {a_2}^2 + 4\,a_2a_1 + {a_1}^2 - 6\,a_2 - 3\,a_1 + 9\right)d^2\\
c_4 &= -4\left({a_2}^4 - 4\,{a_2}^2a_1 + {a_2}^2 + 2\,{a_1}^2 - 8\,a_2 - 2\,a_1 + 3\right)d^2\\
c_3 &= -16\left({a_2}^3{a_1}^2 + 6\,{a_2}^4 + 6\,{a_2}^2{a_1}^2 - 4\,a_2{a_1}^3 + 2\,{a_1}^4 + 4\,{a_2}^3 - 18\,{a_2}^2a_1 - 4\,{a_1}^3 + 10\,{a_2}^2\right.\\
&\quad\left. -10\,a_2a_1 + 10\,{a_1}^2 - 27\,a_2\right)d\\
c_2 &= 16\left(2\,{a_2}^4 + 5\,{a_2}^2{a_1}^2 + {a_1}^4 + 2\,{a_2}^3 - 2\,{a_1}^3 + 8\,{a_2}^2 + {a_1}^2 - 6\,a_2\right)d\\
c_1 &= 64\,{a_2}^2{a_1}^4 + 64\,{a_1}^6 + 320\,{a_2}^3{a_1}^2 + 64\,{a_2}^2{a_1}^3 + 384\,a_2{a_1}^4 - 64\,{a_1}^5 + 320\,{a_2}^4 + 256\,{a_2}^3a_1\\
&\quad +704\,{a_2}^2{a_1}^2 - 384\,a_2{a_1}^3 + 128\,{a_1}^4 + 576\,{a_2}^3 - 384\,{a_2}^2a_1 + 576\,a_2{a_1}^2 + 192\,{a_1}^3 + 1152\,{a_2}^2\\
c_0 &= -64\left({a_2}^2 + a_2a_1 + {a_1}^2 + 2\,a_2 - 2\,a_1 + 4\right)\left({a_2}^2 - a_2a_1 + {a_1}^2\right),
\end{aligned}$$

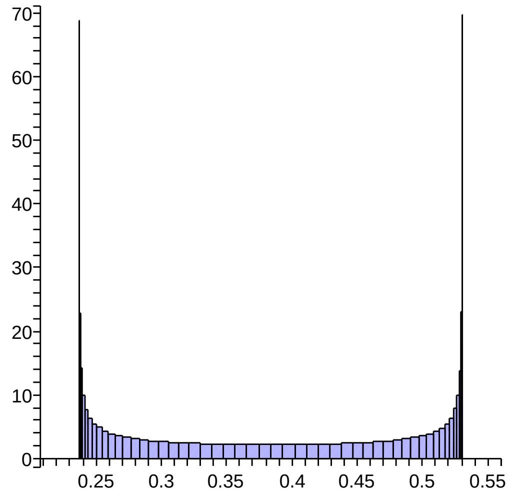
where $d = -4a_1^3 + 4a_2^3 + a_1^2a_2^2 - 18a_1a_2 - 27$ is the discriminant of f.

If $f(x) = x^3 + x^2 + x - 1$, then

$$I = \inf_n K_n = 0.2368.. \quad \text{and} \quad S = \sup_n K_n = 0.5308..$$

are roots of

$$p(x) = \left(11\,x^3 + 11\,x^2 + x - 1\right)\left(121\,x^3 - 143\,x^2 + 55\,x - 7\right)$$

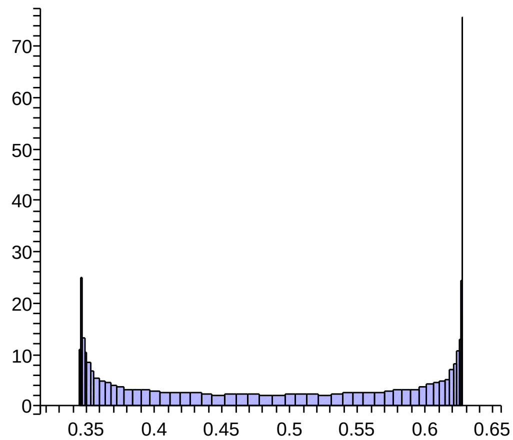


If $f(x) = x^3 + x - 1$, then

$$I = \inf_n K_n = 0.3458.. \quad \text{and} \quad S = \sup_n K_n = 0.6276..$$

are roots of

$$p(x) = 29791\,x^6 - 26908\,x^5 + 11532\,x^4 - 3968\,x^3 - 320\,x + 192$$



Consider a polynomial $f(x) = x^3 + a_2x^2 + a_1x - 1$ in $\mathbb{Z}[x]$ irreducible over $\mathbb{Q}$. Let ξ, ξ_1 and ξ_2 be its roots. Assume that ξ is real and ξ_1, ξ_2 are complex. We also assume that $|\xi| < 1$.

We introduce a sequence of polynomials $P_n(x) = a_n + b_nx + c_nx^2$ in $\mathbb{Z}[x]$, defined by $P_n(x) = x^n \mod f$. Since $f(\xi) = f(\xi_1) = f(\xi_2) = 0$, we have

$$
\begin{aligned}
P_n(\xi) &= \xi^n = a_n + b_n\xi + c_n\xi^2, \\
P_n(\xi_1) &= \xi_1^n = a_n + b_n\xi_1 + c_n\xi_1^2, \\
P_n(\xi_2) &= \xi_2^n = a_n + b_n\xi_2 + c_n\xi_2^2.
\end{aligned}
\tag{1}
$$

In other words,

$$
\begin{pmatrix} \xi^n \\ \xi_1^n \\ \xi_2^n \end{pmatrix} = V \begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix}, \quad \text{where} \quad V = \begin{pmatrix} 1 & \xi & \xi^2 \\ 1 & \xi_1 & \xi_1^2 \\ 1 & \xi_2 & \xi_2^2 \end{pmatrix} \tag{2}
$$

is the Vandermonde matrix. From (2) it follows that

$$
\begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} = V^{-1} \begin{pmatrix} \xi^n \\ \xi_1^n \\ \xi_2^n \end{pmatrix}. \tag{3}
$$

From (2) it follows that

$$\begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} = V^{-1} \begin{pmatrix} \xi^n \\ \xi_1^n \\ \xi_2^n \end{pmatrix}. \tag{3}$$

Since ξ, ξ_1 and ξ_2 are roots of $f(x)$ and the last coefficient of f is -1, it follows that $|\xi\xi_1\xi_2| = 1$. Therefore

$$|\xi|^{-1/2} = |\xi_1| = |\xi_2|, \tag{4}$$

because ξ_1 and ξ_2 are complex conjugates. Applying (4) to (3), we get

$$\begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} = V^{-1}\xi^{-n/2} \begin{pmatrix} \xi^{3n/2} \\ \cos(n\beta) + i\sin(n\beta) \\ \cos(n\beta) - i\sin(n\beta) \end{pmatrix}, \tag{5}$$

where $\beta = \arg(\xi_1)$. Observe that for large n, $\xi^{3n/2}$ is much smaller than $|\cos(n\beta) \pm i\sin(n\beta)|$, therefore

$$\begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} \approx V^{-1}\xi^{-n/2} \begin{pmatrix} 0 \\ \cos(n\beta) + i\sin(n\beta) \\ \cos(n\beta) - i\sin(n\beta) \end{pmatrix},$$

which can be rewritten as

$$\xi^{n/2} \begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} \approx \frac{2i}{D} \begin{pmatrix} \xi^{3/2}\sin((n-1)\beta) - \sin((n-2)\beta) \\ \xi^{-1}\sin((n-2)\beta) - \xi^2\sin(n\beta) \\ \xi\sin(n\beta) - \xi^{-1/2}\sin((n-1)\beta) \end{pmatrix}, \tag{6}$$

where D is the determinant of V.

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$$\xi^{n/2}\begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} \approx \frac{2i}{D}\begin{pmatrix} \xi^{3/2}\sin((n-1)\beta) - \sin((n-2)\beta) \\ \xi^{-1}\sin((n-2)\beta) - \xi^{2}\sin(n\beta) \\ \xi\sin(n\beta) - \xi^{-1/2}\sin((n-1)\beta) \end{pmatrix}, \tag{6}$$

where D is the determinant of V. Put

$$g_1(n) = \xi^{3/2}\sin((n-1)\beta) - \sin((n-2)\beta),$$

$$g_2(n) = \xi^{-1}\sin((n-2)\beta) - \xi^{2}\sin(n\beta),$$

$$g_3(n) = \xi\sin(n\beta) - \xi^{-1/2}\sin((n-1)\beta).$$

From (1) and (6) it follows that

$$(P_n(\xi))^{1/2}a_n \approx \frac{2i}{D}g_1(n), \quad (P_n(\xi))^{1/2}b_n \approx \frac{2i}{D}g_2(n), \quad (P_n(\xi))^{1/2}c_n \approx \frac{2i}{D}g_3(n). \tag{7}$$

We have

$$(P_n(\xi))^{1/2}a_n \approx A_1\sin(n\beta) + B_1\cos(n\beta)$$

$$(P_n(\xi))^{1/2}b_n \approx A_2\sin(n\beta) + B_2\cos(n\beta)$$

$$(P_n(\xi))^{1/2}c_n \approx A_3\sin(n\beta) + B_3\cos(n\beta)$$

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$$(P_n(\xi))^{1/2}a_n \approx A_1\sin(n\beta)+B_1\cos(n\beta)$$

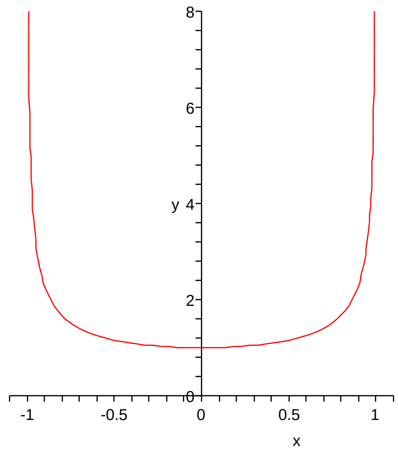
$$(P_n(\xi))^{1/2}b_n \approx A_2\sin(n\beta)+B_2\cos(n\beta)$$

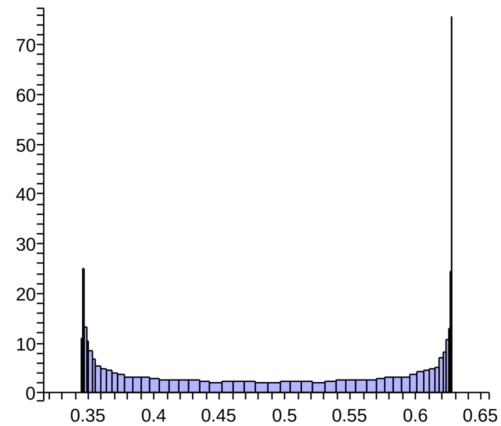
$$(P_n(\xi))^{1/2}c_n \approx A_3\sin(n\beta)+B_3\cos(n\beta)$$

The density function is

$$\varphi(x)=\frac{1}{\sqrt{A_i^2+B_i^2-x^2}},\quad i=1,2,3.$$

If $A_i^2+B_i^2=1$,





Put

$$g_1(n) = \xi^{3/2}\sin((n-1)\beta) - \sin((n-2)\beta),$$

$$g_2(n) = \xi^{-1}\sin((n-2)\beta) - \xi^2\sin(n\beta),$$

$$g_3(n) = \xi\sin(n\beta) - \xi^{-1/2}\sin((n-1)\beta).$$

From (1) and (6) it follows that

$$(P_n(\xi))^{1/2}a_n \approx \frac{2i}{D}g_1(n), \quad (P_n(\xi))^{1/2}b_n \approx \frac{2i}{D}g_2(n), \quad (P_n(\xi))^{1/2}c_n \approx \frac{2i}{D}g_3(n). \tag{7}$$

To show that the points $(g_1(n), g_2(n), g_3(n))$ trace an ellipse in 3-dimensional space, we note that

$$g_1(n) + \xi g_2(n) + \xi^2 g_3(n) = 0$$

and

$$-\frac{g_1^2(n)}{a^2} + \frac{g_2^2(n)}{b^2} + \frac{g_3^2(n)}{c^2} = 1,$$

where

$$a^2 = \frac{\xi^{3/2}\sin^2\beta(\xi^3 + 1 - 2\xi^{3/2}\cos\beta)}{\xi^{3/2} + \cos\beta},$$

$$b^2 = -\frac{\sin^2\beta(\xi^3 + 1 - 2\xi^{3/2}\cos\beta)}{\xi^{1/2}\cos\beta},$$

$$c^2 = \frac{\sin^2\beta(\xi^3 + 1 - 2\xi^{3/2}\cos\beta)}{\xi(1 + \xi^{3/2}\cos\beta)}.$$

Put

$$g_1(n) = \xi^{3/2}\sin((n-1)\beta) - \sin((n-2)\beta),$$

$$g_2(n) = \xi^{-1}\sin((n-2)\beta) - \xi^2\sin(n\beta),$$

$$g_3(n) = \xi\sin(n\beta) - \xi^{-1/2}\sin((n-1)\beta).$$

From (1) and (6) it follows that

$$(P_n(\xi))^{1/2}a_n \approx \frac{2i}{D}g_1(n), \quad (P_n(\xi))^{1/2}b_n \approx \frac{2i}{D}g_2(n), \quad (P_n(\xi))^{1/2}c_n \approx \frac{2i}{D}g_3(n). \tag{7}$$

We now find

$$\inf_n |P_n(\xi)|(a_n^2 + b_n^2 + c_n^2) \quad \text{and} \quad \sup_n |P_n(\xi)|(a_n^2 + b_n^2 + c_n^2).$$

By (7) we have

$$\inf_n |P_n(\xi)|(a_n^2 + b_n^2 + c_n^2) = -\inf_n 4(g_1^2(n) + g_2^2(n) + g_3^2(n))/D^2$$

and

$$\sup_n |P_n(\xi)|(a_n^2 + b_n^2 + c_n^2) = -\sup_n 4(g_1^2(n) + g_2^2(n) + g_3^2(n))/D^2$$

One can see that

$$(g_1(n))^2 + (g_2(n))^2 + (g_3(n))^2 = A\sin(n\beta)\cos(n\beta) + B\cos^2(n\beta) + C,$$

where $A = A(\xi,\beta), B = B(\xi,\beta), C = C(\xi,\beta)$.

One can see that

$$(g_1(n))^2 + (g_2(n))^2 + (g_3(n))^2 = A\sin(n\beta)\cos(n\beta) + B\cos^2(n\beta) + C,$$

where $A = A(\xi,\beta), B = B(\xi,\beta), C = C(\xi,\beta)$. One can show that the function

$$f(x) = Ax\sqrt{1-x^2} + Bx^2 + C$$

attains its minimum(maximum) values at

$$x_{min} = -\frac{1}{\sqrt{2}}\sqrt{1 - \frac{B}{\sqrt{A^2+B^2}}} \quad \text{and} \quad x_{max} = \frac{1}{\sqrt{2}}\sqrt{1 + \frac{B}{\sqrt{A^2+B^2}}}$$

and these values are

$$f_{min} = \frac{B - \sqrt{A^2+B^2}}{2} + C \quad \text{and} \quad f_{max} = \frac{B + \sqrt{A^2+B^2}}{2} + C. \tag{8}$$

Using explicit expressions for $A(\xi,\beta), B(\xi,\beta)$ and $C(\xi,\beta)$ in (8) [**here the algebra becomes really ugly and I don't know how to make it more elegant yet**], we deduce that $-\inf_n 4(g_1^2(n) + g_2^2(n) + g_3^2(n))/D^2$ and $-\sup_n 4(g_1^2(n) + g_2^2(n) + g_3^2(n))/D^2$ are roots of the following polynomial:

$$p(x) = c_6x^6 + \ldots + c_2x^2 + c_1x + c_0,$$

Using explicit expressions for $A(\xi,\beta), B(\xi,\beta)$ and $C(\xi,\beta)$ in (8) [**here the algebra becomes really ugly and I don't know how to make it more elegant yet**], we deduce that $-\inf_n 4(g_1^2(n)+g_2^2(n)+g_3^2(n))/D^2$ and $-\sup_n 4(g_1^2(n)+g_2^2(n)+g_3^2(n))/D^2$ are roots of the following polynomial:

$$p(x) = c_6x^6 + \ldots + c_2x^2 + c_1x + c_0,$$

with

$$c_6 = d^3$$

$$c_5 = 4\left({a_2}^4 - {a_2}^3 - 4\,{a_2}^2a_1 + {a_2}^2 + 4\,a_2a_1 + {a_1}^2 - 6\,a_2 - 3\,a_1 + 9\right)d^2$$

$$c_4 = -4\left({a_2}^4 - 4\,{a_2}^2a_1 + {a_2}^2 + 2\,{a_1}^2 - 8\,a_2 - 2\,a_1 + 3\right)d^2$$

$$c_3 = -16\left({a_2}^3{a_1}^2 + 6\,{a_2}^4 + 6\,{a_2}^2{a_1}^2 - 4\,a_2{a_1}^3 + 2\,{a_1}^4 + 4\,{a_2}^3 - 18\,{a_2}^2a_1 - 4\,{a_1}^3 + 10\,{a_2}^2 - 10\,a_2a_1 + 10\,{a_1}^2 - 27\,a_2\right)d$$

$$c_2 = 16\left(2\,{a_2}^4 + 5\,{a_2}^2{a_1}^2 + {a_1}^4 + 2\,{a_2}^3 - 2\,{a_1}^3 + 8\,{a_2}^2 + {a_1}^2 - 6\,a_2\right)d$$

$$c_1 = 64\,{a_2}^2{a_1}^4 + 64\,{a_1}^6 + 320\,{a_2}^3{a_1}^2 + 64\,{a_2}^2{a_1}^3 + 384\,a_2{a_1}^4 - 64\,{a_1}^5 + 320\,{a_2}^4 + 256\,{a_2}^3a_1 + 704\,{a_2}^2{a_1}^2 - 384\,a_2{a_1}^3 + 128\,{a_1}^4 + 576\,{a_2}^3 - 384\,{a_2}^2a_1 + 576\,a_2{a_1}^2 + 192\,{a_1}^3 + 1152\,{a_2}^2$$

$$c_0 = -64\left({a_2}^2 + a_2a_1 + {a_1}^2 + 2\,a_2 - 2\,a_1 + 4\right)\left({a_2}^2 - a_2a_1 + {a_1}^2\right),$$

where a_1 and a_2 are coefficients of $f(x) = x^3 + a_2x^2 + a_1x - 1$ and

$$d = -4a_1^3 + 4a_2^3 + a_1^2a_2^2 - 18a_1a_2 - 27$$

is the discriminant of f.

For example,

if $f(x) = x^3 + x^2 + x - 1$, then $p(x) = 64\ (11\,x^3 + 11\,x^2 + x - 1)\,(121\,x^3 - 143\,x^2 + 55\,x - 7)$

if $f(x) = x^3 + x^2 - 1$, then $p(x) = 12167\,x^6 - 8464\,x^5 - 6348\,x^4 + 2576\,x^3 + 2208\,x^2 - 2048\,x + 448$

if $f(x) = x^3 + x - 1$, then $p(x) = 29791\,x^6 - 26908\,x^5 + 11532\,x^4 - 3968\,x^3 - 320\,x + 192$

if $f(x) = x^3 - x^2 + 2x - 1$, then $p(x) = 12167\,x^6 + 19044\,x^4 - 42320\,x^3 + 13984\,x^2 - 2944\,x + 448$