On approximation of real numbers by algebraic numbers of bounded degree

BY

K. I. Tsishchanka

THEOREM **(Dirichlet, 1842): For any real irrational number ξ there exist infinitely many rational numbers p/q such that**

$$\left|\xi - \frac{p}{q}\right| < \frac{1}{q^2}$$

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Example: If $\xi = \pi$, then

$$\left|\xi - \frac{22}{7}\right| < \frac{1}{7^2}$$

$$\left|\xi - \frac{333}{106}\right| < \frac{1}{106^2}$$

$$\left|\xi - \frac{355}{113}\right| < \frac{1}{113^2}$$

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EXAMPLE: **Suppose $\xi = \frac{m}{n}$. Then**

$$\left|\xi - \frac{m}{n}\right| = \left|\xi - \frac{2m}{2n}\right| = \left|\xi - \frac{3m}{3n}\right| = \left|\xi - \frac{4m}{4n}\right| = 0 < \frac{1}{q^2}.$$

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<u>REMARK</u>: Note that if $\xi = \frac{m}{n}$ and $\frac{p}{q}$ is irreducible with $\frac{p}{q} \neq \frac{m}{n}$, then

$$\left|\xi - \frac{p}{q}\right| = \left|\frac{m}{n} - \frac{p}{q}\right| = \left|\frac{mq - pn}{nq}\right| \geq \frac{1}{nq} > \frac{1}{q^2}$$

for a sufficiently large q.

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<u>Theorem</u> (Hurwitz): Let $\xi = (\sqrt{5} - 1)/2$. Then for any $c < 1/\sqrt{5}$ the inequality

$$\left|\xi - \frac{p}{q}\right| < \frac{c}{q^2}$$

has only finitely many solutions.

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Example: **Let $\xi = \frac{1}{2} + \frac{1}{2^n}$. Then**

$$\left|\xi - \frac{1}{2}\right| = \frac{1}{2^n}$$

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<u>Example</u>: Let $\xi = \frac{1}{2} + \frac{1}{2^3}$. Then

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<u>Theorem</u> (Liouville): Let $\xi = \sum_{k=1}^{\infty} \frac{1}{10^{k!}}$. Then for any $\ell \geq 2$ there exist infinitely many rational numbers p/q such that

$$\left|\xi - \frac{p}{q}\right| < \frac{1}{q^\ell}$$

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THEOREM (Liouville): Suppose $\alpha \in \mathbb{R}$ is a root of a nonzero irreducible polynomial P of degree $n \geq 2$. Then for any $c < 1/|P'(\alpha)|$ the inequality

$$\left|\alpha - \frac{p}{q}\right| < \frac{c}{q^n}$$

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<u>Theorem</u> (Roth, 1958): For algebraic α of degree $n \geq 2$,

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^{2+\epsilon}}$$

with $\epsilon > 0$, has only finitely many solutions.

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Davenport presented Roth with the Fields Medal at the International Congress in Edinburgh in 1958. Speaking of Roth's solution to this problem of approximating algebraic numbers Davenport:

The achievement is one that speaks for itself: it closes a chapter, and a new chapter is now opened. Roth's theorem settles a question which is both of a fundamental nature and of extreme difficulty. It will stand as a landmark in mathematics for as long as mathematics is cultivated.

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THEOREM (Davenport-Schmidt, 1967): For any real number $\xi \notin \mathbb{A}_2$ there exist infinitely many algebraic numbers $\alpha \in \mathbb{A}_2$ such that

$$|\xi - \alpha| < \frac{160}{9} H(\alpha)^{-3}$$

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Conjecture: For any real number $\xi \notin \mathbb{A}_n$ there exist infinitely many algebraic numbers $\alpha \in \mathbb{A}_n$ such that

$$|\xi - \alpha| \ll H(\alpha)^{-n-1}$$

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<u>CONJECTURE</u>: For any real number $\xi \notin \mathbb{A}_3$ there exist infinitely many algebraic numbers $\alpha \in \mathbb{A}_3$ such that

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Theorem (Wirsing, 1961): For any real number $\xi \notin \mathbb{A}_3$ there exist infinitely many algebraic numbers $\alpha \in \mathbb{A}_3$ such that

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THEOREM (Bernik-T., 1993): For any real number $\xi \notin \mathbb{A}_3$ there exist infinitely many algebraic numbers $\alpha \in \mathbb{A}_3$ such that

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3	0.14159
$\frac{22}{7}$	0.0619
$\frac{333}{106}$	0.93505
$\frac{355}{113}$	0.00340
$\frac{103993}{33102}$	0.63321
$\frac{104348}{33215}$	0.36586
$\frac{208341}{66317}$	0.53811
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<u>Theorem</u> (Dirichlet): For any real irrational number ξ there exist infinitely many rational numbers p/q such that

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Theorem (Hurwitz): For any real irrational number ξ there exist infinitely many rational numbers p/q such that

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$$p_{i+1} = p_i + p_{i-1}$$

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$$|\xi - \alpha| < \frac{160}{9} H(\alpha)^{-3}$$

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Theorem (T., 2007): For any real number $\xi \notin \mathbb{A}_3$ there exist infinitely many algebraic numbers $\alpha \in \mathbb{A}_3$ such that

$$|\xi - \alpha| \ll H(\alpha)^{-3.73\ldots}$$

<u>Theorem</u> **(Hurwitz): For any real irrational number ξ there exist infinitely many i such that**

$$|q_i\xi - p_i|\, q_i < \frac{1}{\sqrt{5}}$$

Theorem (Hurwitz): **For any real irrational number ξ there exist infinitely many i such that**

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THEOREM (Hurwitz): **For any real irrational number ξ there exist infinitely many i such that**

$$|P_i(\xi)| < \frac{1}{\sqrt{5}} q_i^{-1}, \quad \text{where} \quad P_i(x) = q_i x - p_i$$

<u>Theorem</u> **(Hurwitz): For any real irrational number ξ there exist infinitely many integer polynomials $P(x)$ of first degree such that**

$$|P(\xi)| < \frac{1}{\sqrt{5}} q^{-1}, \quad \text{where} \quad P(x) = qx - p$$

THEOREM (Dirichlet): **For any real irrational number ξ there exist infinitely many integer polynomials $P(x)$ of first degree such that**

$$|P(\xi)| \ll \max(|p|, |q|)^{-1}, \quad \text{where} \quad P(x) = qx - p$$

Theorem (Dirichlet): For any real irrational number $\boldsymbol{\xi}$ there exist infinitely many integer polynomials $\boldsymbol{P(x)}$ of first degree such that

$$|P(\xi)| \ll \lceil P \rceil^{-1}$$

<u>Theorem</u> (Dirichlet): For any real number $\xi \notin \mathbb{A}_n$ there exist infinitely many integer polynomials $P(x)$ of degree $\leq n$ such that

$$|P(\xi)| \ll \lceil P \rceil^{-n}$$

$$\begin{array}{ccc}
\left|\xi - \dfrac{p}{q}\right| < q^{-2} & \longleftrightarrow & |q\xi - p| < q^{-1} \\
 & & \downarrow \\
|\xi - \alpha| \ll H(\alpha)^{-n-1} & & |P(\xi)| \ll \lceil P \rceil^{-n}
\end{array}$$

<u>Theorem</u> **(Dirichlet): For any real number $\boldsymbol{\xi} \notin \mathbb{A}_n$ there exist infinitely many integer polynomials $\boldsymbol{P(x)}$ of degree $\leq \boldsymbol{n}$ such that**

$$|\boldsymbol{P}(\boldsymbol{\xi})| \ll \lceil \boldsymbol{P} \rceil^{-n}$$

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$$|P(\xi)| \ll \lceil P \rceil^{-n}$$

Theorem (Dirichlet): For any real number $\xi \notin \mathbb{A}_n$ there exist infinitely many integer numbers $q, p_1, \ldots, p_n$ such that

$$\begin{cases} |q\xi + p_1| < q^{-1-\frac{1}{n}} \\ \ldots \\ |q\xi^n + p_n| < q^{-1-\frac{1}{n}} \end{cases}$$

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Theorem (Dirichlet): For any real number $\xi \notin \mathbb{A}_n$ there exist infinitely many integer numbers $q, p_1, \ldots, p_n$ such that

$$\begin{cases} |q\xi + p_1| < q^{-1-\frac{1}{n}} \\ \ldots \\ |q\xi^n + p_n| < q^{-1-\frac{1}{n}} \end{cases}$$

Example: For any real number $\xi \notin \mathbb{A}_1$ there exist infinitely many integer numbers q, p_1 such that

$$\left\{ |q\xi + p_1| < q^{-1-\frac{1}{1}} \right.$$

<u>Theorem</u> (Dirichlet): For any real number $\xi \notin \mathbb{A}_n$ there exist infinitely many integer polynomials $P(x)$ of degree $\leq n$ such that

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<u>Example</u>: For any real number $\xi \notin \mathbb{A}_2$ there exist infinitely many integer numbers q, p_1, p_2 such that

$$\begin{cases} |q\xi + p_1| < q^{-1-\frac{1}{2}} \\ \\ \left|q\xi^2 + p_2\right| < q^{-1-\frac{1}{2}} \end{cases}$$

Theorem (Dirichlet): For any real number $\xi \notin \mathbb{A}_n$ there exist infinitely many integer polynomials $P(x)$ of degree $\leq n$ such that

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Theorem (Dirichlet): For any real irrational numbers $\xi_1, \ldots, \xi_n$ there exist infinitely many integer numbers $q, p_1, \ldots, p_n$ such that

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Minkowski (1896), Voronoi (1896), Jacobi-Perron (1907), Furtwangler (1928), Brun (1957), Szekeres (1970), Brentjes (1981), Lenstra-Lenstra-Lovasz (1982), Lagarias (1994), Rossner-Schnorr (1996), Vaughan-Clarkson (1996), Arnold (1998), Briggs (2000), etc.

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In any deeper investigation of the problem of simultaneous approximation we are greatly handicapped by the absence of a full analogue of the continued fraction process. There are several analogues, but they all suffer from one of two defects: either they give much poorer approximations than we known to exist, or they involve a series of operations that can be carried out on given $\xi_1, \xi_2, \ldots, \xi_n$ *but cannot be used to define* $\xi_1, \xi_2, \ldots, \xi_n$*, because we do not know the limitations to which any such sequence is subject.* **(Davenport, 1956)**

__Theorem__ **(Dirichlet): For any real number ξ which is not rational nor quadratic there exist infinitely many integer numbers a, b, and c such that**

$$\left|a\xi^2 + b\xi + c\right| \ll \max(|a|, |b|, |c|)^{-2}$$

__Theorem__ **(Dirichlet): For any real irrational number ξ there exist infinitely many integer numbers q, p_1, and p_2 such that**

$$\begin{cases} |q\xi + p_1| < q^{-1/2} \\ \left|q\xi^2 + p_2\right| < q^{-1/2} \end{cases}$$

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$$|q\xi + p| < q^{-1}$$

$$\xi = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}}$$

$$p_{i+1} = a_i p_i + p_{i-1}$$
$$q_{i+1} = a_i q_i + q_{i-1}$$

THEOREM (Dirichlet): For any real number ξ which is not rational nor quadratic there exist infinitely many integer numbers a, b, and c such that

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$$\xi = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}} \qquad (p_{i+1}, q_{i+1}) = a_i(p_i, q_i) + (p_{i-1}, q_{i-1})$$

Let $\xi = 0.6823...$ be a root of $x^3 + x - 1$.

Algorithm:

1. Put $P_1 = 1,\ P_2 = x,\ P_3 = x^2$.

2. Consider

$$aP_1 + bP_2 + cP_3$$

with

$$-N \le a, b, c \le N,$$

where N is sufficiently large.

3. Choose a polynomial P_4 with the smallest height such that

$$|P_4(\xi)| < |P_3(\xi)|.$$

Note that $a = 1,\ b = -1,\ c = -1$.

4	$-x^2 - x + 1$	$[1, -1, -1]$	0.1478990
5	$-2x^2 + 1$	$[1, -1, 1]$	0.2754301
6	$-2x^2 - 3x + 3$	$[1, 3, 0]$	0.1968671
7	$-5x^2 - x + 3$	$[1, 2, 0]$	0.2545991
8	$-8x^2 + 4x + 1$	$[1, -2, 2]$	0.3034471
9	$-4x^2 - 9x + 8$	$[1, 2, -1]$	0.2620481
10	$-12x^2 - 5x + 9$	$[0, 1, 1]$	0.2168926

Let $\xi = 0.6823...$ be a root of $x^3 + x - 1$.

Algorithm:

1. Put $P_2 = x, P_3 = x^2, P_4 = -x^2 - x + 1$.

2. Consider

$$aP_2 + bP_3 + cP_4$$

with

$$-N \leq a, b, c \leq N,$$

where N is sufficiently large.

3. Choose a polynomial P_5 with the smallest height such that

$$|P_5(\xi)| < |P_4(\xi)|.$$

Note that $a = 1,\ b = -1,\ c = 1$.

4	$-x^2 - x + 1$	$[1, -1, -1]$	0.1478990
5	$-2\,x^2 + 1$	$[1, -1, 1]$	0.2754301
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10	$-12x^2 - 5x + 9$	$[0, 1, 1]$	0.2168926
11	$-28x^2 - 19x + 26$	$[1, 2, 1]$	0.1746480
12	$-180x^2 - 119x + 165$	$[10, 23, 8]$	5.495255

Let $\xi = 0.6823...$ be a root of $x^3 + x - 1$.

Algorithm:

1. Put $P_7 = -5\,x^2 - x + 3,$
$P_8 = -8\,x^2 + 4\,x + 1,$
$P_9 = -4\,x^2 - 9\,x + 8.$

2. Consider

$$aP_7 + bP_8 + cP_9$$

with

$$-N \le a, b, c \le N,$$

where N is sufficiently large.

3. Choose a polynomial P_{11} with the smallest height such that

$$|P_{11}(\xi)| < |P_{10}(\xi)|.$$

4	$-x^2 - x + 1$	$[1, -1, -1]$	0.1478990
5	$-2\,x^2 + 1$	$[1, -1, 1]$	0.2754301
6	$-2\,x^2 - 3\,x + 3$	$[1, 3, 0]$	0.1968671
7	$-5\,x^2 - x + 3$	$[1, 2, 0]$	0.2545991
8	$-8\,x^2 + 4\,x + 1$	$[1, -2, 2]$	0.3034471
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Let $\xi = 0.6823...$ be a root of x^3+x-1.

Algorithm:

1. Put $P_7 = -5\,x^2 - x + 3,$
$P_8 = -8\,x^2 + 4\,x + 1,$
$P_9 = -4\,x^2 - 9\,x + 8.$

2. Consider

$$aP_7 + bP_8 + cP_9$$

with

$$-N \le a, b, c \le N,$$

where N is sufficiently large.

3. Choose a polynomial P_{11} with the smallest height such that

$$|P_{11}(\xi)| < |P_{10}(\xi)|.$$

Note that $a = 1,\ b = 2,\ c = 0$.

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10	$-12\,x^2 - 5\,x + 9$	$[0, 1, 1]$	0.2168926
11	$-21\,x^2 + 7\,x + 5$	$[1, 2, 0]$	0.3092480
12	$-7\,x^2 - 26\,x + 21$	$[1, 2, -1]$	0.3234507
13	$-28\,x^2 - 19\,x + 26$	$[0, 1, 1]$	0.1746480
14	$-54\,x^2 + 9\,x + 19$	$[1, 2, 0]$	0.3024274
15	$-63\,x^2 - 64\,x + 73$	$[1, 2, 0]$	0.1755733
16	$-136\,x^2 - x + 64$	$[1, 2, 0]$	0.2837114
17	$-200\,x^2 + 135\,x + 1$	$[1, -2, 2]$	0.2856572
18	$-135\,x^2 - 201\,x + 200$	$[1, 2, -1]$	0.1968658
19	$-335\,x^2 - 66\,x + 201$	$[0, 1, 1]$	0.2545974
20	$-536\,x^2 + 269\,x + 66$	$[1, 2, 0]$	0.3034450
21	$-269\,x^2 - 602\,x + 536$	$[1, 2, -1]$	0.2611780
22	$-805\,x^2 - 333\,x + 602$	$[0, 1, 1]$	0.2174310
23	$-1407\,x^2 + 472\,x + 333$	$[1, 2, 0]$	0.3092459

Let $\xi = 0.6180...$ be a root of $x^2 + x - 1$.

1	-1		-1
2	x		0.6180339887
3	$x - 1$		0.3819660113
4	$2\,x - 1$	$[1, 1, 1]$	0.4721359550
5	$3\,x - 2$	$[1, 1, 1]$	0.4376941013
6	$5\,x - 3$	$[1, 1, 1]$	0.4508497187
7	$8\,x - 5$	$[1, 1, 1]$	0.4458247200
8	$13\,x - 8$	$[1, 1, 1]$	0.4477440987
9	$21\,x - 13$	$[1, 1, 1]$	0.4470109613
10	$34\,x - 21$	$[1, 1, 1]$	0.4472909949
11	$55\,x - 34$	$[1, 1, 1]$	0.4471840316
12	$89\,x - 55$	$[1, 1, 1]$	0.4472248879
13	$144\,x - 89$	$[1, 1, 1]$	0.4472092822
14	$233\,x - 144$	$[1, 1, 1]$	0.4472152430
15	$377\,x - 233$	$[1, 1, 1]$	0.4472129662
16	$610\,x - 377$	$[1, 1, 1]$	0.4472138359
17	$987\,x - 610$	$[1, 1, 1]$	0.4472135037
18	$1597\,x - 987$	$[1, 1, 1]$	0.4472136306
19	$2584\,x - 1597$	$[1, 1, 1]$	0.4472135821

Let $\xi = 0.6823...$ be a root of $x^3 + x - 1$.

Algorithm:

1. Put $P_7 = -5\,x^2 - x + 3,$
$P_8 = -8\,x^2 + 4\,x + 1,$
$P_9 = -4\,x^2 - 9\,x + 8.$

2. Consider

$$aP_7 + bP_8 + cP_9$$

with

$$-N \le a, b, c \le N,$$

where N is sufficiently large.

3. Choose a polynomial P_{11} with the smallest height such that

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Note that $a = 1,\ b = 2,\ c = 0.$

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20	$-536\,x^2 + 269\,x + 66$	$[1, 2, 0]$	0.3034450
21	$-269\,x^2 - 602\,x + 536$	$[1, 2, -1]$	0.2611780
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<u>Theorem</u> (Lagarias, 1982): For any given norm $\|\cdot\|$ on $\mathbb{R}^n$ with $n \geq 2$ there exists an $\alpha \in \mathbb{R}^n$ with $\dim_Q[1, \alpha_1, \ldots, \alpha_n] = n+1$ such that for any positive integer L there exists an integer k (depending on L) such that the best approximation determinants of α with respect to $\|\cdot\|$ have the property

$$D_k = D_{k+1} = \ldots = D_{k+L} = 0.$$

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23	$-1407x^2 + 472x + 333$	$[1, 2, 0]$	0.3092459

24	$-472\,x^2-1740\,x+1407$	$[1,2,-1]$	0.3227062
25	$1879\,x^2+1268\,x-1740$	$[0,-1,-1]$	0.1752058
26	$-3619\,x^2+611\,x+1268$	$[1,2,0]$	0.3025926
27	$-4230\,x^2-4276\,x+4887$	$[1,-2,0]$	0.1752851
28	$9117\,x^2+46\,x-4276$	$[1,-2,0]$	0.2840210
29	$-13393\,x^2+9071\,x+46$	$[1,-2,-2]$	0.2853569
30	$-9071\,x^2-13439\,x+13393$	$[1,-2,-1]$	0.1960467
31	$-22464\,x^2-4368\,x+13439$	$[0,1,1]$	0.2550269
32	$35903\,x^2-18096\,x-4368$	$[1,-2,0]$	0.3032909
33	$-18096\,x^2-40271\,x+35903$	$[1,2,1]$	0.2603610
34	$-53999\,x^2-22175\,x+40271$	$[0,-1,1]$	0.2179459
35	$-94270\,x^2+31824\,x+22175$	$[1,-2,0]$	0.3092504
36	$-31824\,x^2-116445\,x+94270$	$[1,2,-1]$	0.3219571
37	$126094\,x^2+84621\,x-116445$	$[0,-1,-1]$	0.1757646
38	$-242539\,x^2+41473\,x+84621$	$[1,2,0]$	0.3027553
39	$-284012\,x^2-285687\,x+327160$	$[1,-2,0]$	0.1749961
40	$611172\,x^2+1675\,x-285687$	$[1,-2,0]$	0.2843288
41	$-896859\,x^2+609497\,x+1675$	$[1,-2,-2]$	0.2850549
42	$-609497\,x^2-898534\,x+896859$	$[1,-2,-1]$	0.1952281
43	$-1506356\,x^2-289037\,x+898534$	$[0,1,1]$	0.2554551

44	$2404890\,x^2 - 1217319\,x - 289037$	$[1,-2,0]$	0.3031349
45	$-1217319\,x^2 - 2693927\,x + 2404890$	$[1,2,1]$	0.2595435
46	$-3622209\,x^2 - 1476608\,x + 2693927$	$[0,-1,1]$	0.2184599
47	$-6316136\,x^2 + 2145601\,x + 1476608$	$[1,-2,0]$	0.3092529
48	$-2145601\,x^2 - 7792744\,x + 6316136$	$[1,2,-1]$	0.3212068
49	$-8461737\,x^2 - 5647143\,x + 7792744$	$[0,1,1]$	0.1763230
50	$-16254481\,x^2 + 2814594\,x + 5647143$	$[1,2,0]$	0.3029161
51	$-19069075\,x^2 - 19087030\,x + 21901624$	$[1,2,0]$	0.1747061
52	$-40970699\,x^2 - 17955\,x + 19087030$	$[1,2,0]$	0.2846350
53	$-60057729\,x^2 + 40952744\,x + 17955$	$[1,-2,2]$	0.2847511
54	$-40952744\,x^2 - 60075684\,x + 60057729$	$[1,2,-1]$	0.1944098
55	$-101010473\,x^2 - 19122940\,x + 60075684$	$[0,1,1]$	0.2558820
56	$-161086157\,x^2 + 81887533\,x + 19122940$	$[1,2,0]$	0.3029768
57	$-81887533\,x^2 - 180209097\,x + 161086157$	$[1,2,-1]$	0.2587256
58	$242973690\,x^2 + 98321564\,x - 180209097$	$[0,-1,-1]$	0.2189731
59	$-423182787\,x^2 + 144652126\,x + 98321564$	$[1,2,0]$	0.3092533
60	$-144652126\,x^2 - 521504351\,x + 423182787$	$[1,-2,-1]$	0.3204552
61	$-567834913\,x^2 - 376852225\,x + 521504351$	$[0,1,1]$	0.1768812
62	$1089339264\,x^2 - 190982688\,x - 376852225$	$[1,-2,0]$	0.3030749
63	$-1280321952\,x^2 - 1275208801\,x + 1466191489$	$[1,2,0]$	0.1744152

Let $\xi = 0.6180...$ be a root of $x^2 + x - 1$.

1	-1		-1
2	x		0.6180339887
3	$x - 1$		0.3819660113
4	$2\,x - 1$	$[1, 1, 1]$	0.4721359550
5	$3\,x - 2$	$[1, 1, 1]$	0.4376941013
6	$5\,x - 3$	$[1, 1, 1]$	0.4508497187
7	$8\,x - 5$	$[1, 1, 1]$	0.4458247200
8	$13\,x - 8$	$[1, 1, 1]$	0.4477440987
9	$21\,x - 13$	$[1, 1, 1]$	0.4470109613
10	$34\,x - 21$	$[1, 1, 1]$	0.4472909949
11	$55\,x - 34$	$[1, 1, 1]$	0.4471840316
12	$89\,x - 55$	$[1, 1, 1]$	0.4472248879
13	$144\,x - 89$	$[1, 1, 1]$	0.4472092822
14	$233\,x - 144$	$[1, 1, 1]$	0.4472152430
15	$377\,x - 233$	$[1, 1, 1]$	0.4472129662
16	$610\,x - 377$	$[1, 1, 1]$	0.4472138359
17	$987\,x - 610$	$[1, 1, 1]$	0.4472135037
18	$1597\,x - 987$	$[1, 1, 1]$	0.4472136306
19	$2584\,x - 1597$	$[1, 1, 1]$	0.4472135821
20	$4181\,x - 2584$	$[1, 1, 1]$	0.4472136006

Let $\xi = 0.4307...$ be a root of $65x^2 + 239x - 115$.

1	-1		-1
2	x		0.4307170789
3	$2x - 1$		0.2771316843
4	$7x - 3$	$[1, 3, 1]$	0.1051368676
5	$65x - 28$	$[1, 9, 1]$	0.2203415150
6	$267x - 115$	$[1, 4, 1]$	0.3898399376
7	$599x - 258$	$[1, 2, 1]$	0.2813623499
8	$2064x - 889$	$[1, 3, 1]$	0.1050871615
9	$19175x - 8259$	$[1, 9, 1]$	0.2203440470
10	$78764x - 33925$	$[1, 4, 1]$	0.3898394678
11	$176703x - 76109$	$[1, 2, 1]$	0.2813623985
12	$608873x - 262252$	$[1, 3, 1]$	0.1050871609
13	$5656560x - 2436377$	$[1, 9, 1]$	0.2203440470
14	$23235113x - 10007760$	$[1, 4, 1]$	0.3898394678
15	$52126786x - 22451897$	$[1, 2, 1]$	0.2813623985
16	$179615471x - 77363451$	$[1, 3, 1]$	0.1050871609
17	$1668666025x - 718722956$	$[1, 9, 1]$	0.2203440470
18	$6854279571x - 2952255275$	$[1, 4, 1]$	0.3898394678
19	$15377225167x - 6623233506$	$[1, 2, 1]$	0.2813623985
20	$52985955072x - 22821955793$	$[1, 3, 1]$	0.1050871609

Theorem (Euler-Lagrange): **Let α be real irrational. The continued fraction of α is periodic if and only if α is quadratic.**

4	$-x^2 - x + 1$	$[1, -1, -1]$	0.1478990
5	$-2x^2 + 1$	$[1, -1, 1]$	0.2754301
6	$-2x^2 - 3x + 3$	$[1, 3, 0]$	0.1968671
7	$-5x^2 - x + 3$	$[1, 2, 0]$	0.2545991
8	$-8x^2 + 4x + 1$	$[1, -2, 2]$	0.3034471
9	$-4x^2 - 9x + 8$	$[1, 2, -1]$	0.2620481
10	$-12x^2 - 5x + 9$	$[0, 1, 1]$	0.2168926
11	$-21x^2 + 7x + 5$	$[1, 2, 0]$	0.3092480
12	$-7x^2 - 26x + 21$	$[1, 2, -1]$	0.3234507
13	$-28x^2 - 19x + 26$	$[0, 1, 1]$	0.1746480
14	$-54x^2 + 9x + 19$	$[1, 2, 0]$	0.3024274
15	$-63x^2 - 64x + 73$	$[1, 2, 0]$	0.1755733
16	$-136x^2 - x + 64$	$[1, 2, 0]$	0.2837114
17	$-200x^2 + 135x + 1$	$[1, -2, 2]$	0.2856572
18	$-135x^2 - 201x + 200$	$[1, 2, -1]$	0.1968658
19	$-335x^2 - 66x + 201$	$[0, 1, 1]$	0.2545974
20	$-536x^2 + 269x + 66$	$[1, 2, 0]$	0.3034450
21	$-269x^2 - 602x + 536$	$[1, 2, -1]$	0.2611780
22	$-805x^2 - 333x + 602$	$[0, 1, 1]$	0.2174310
23	$-1407x^2 + 472x + 333$	$[1, 2, 0]$	0.3092459

24	$-472\,x^2 - 1740\,x + 1407$	$[1, 2, -1]$	0.3227062
25	$1879\,x^2 + 1268\,x - 1740$	$[0, -1, -1]$	0.1752058
26	$-3619\,x^2 + 611\,x + 1268$	$[1, 2, 0]$	0.3025926
27	$-4230\,x^2 - 4276\,x + 4887$	$[1, -2, 0]$	0.1752851
28	$9117\,x^2 + 46\,x - 4276$	$[1, -2, 0]$	0.2840210
29	$-13393\,x^2 + 9071\,x + 46$	$[1, -2, -2]$	0.2853569
30	$-9071\,x^2 - 13439\,x + 13393$	$[1, -2, -1]$	0.1960467
31	$-22464\,x^2 - 4368\,x + 13439$	$[0, 1, 1]$	0.2550269
32	$35903\,x^2 - 18096\,x - 4368$	$[1, -2, 0]$	0.3032909
33	$-18096\,x^2 - 40271\,x + 35903$	$[1, 2, 1]$	0.2603610
34	$-53999\,x^2 - 22175\,x + 40271$	$[0, -1, 1]$	0.2179459
35	$-94270\,x^2 + 31824\,x + 22175$	$[1, -2, 0]$	0.3092504
36	$-31824\,x^2 - 116445\,x + 94270$	$[1, 2, -1]$	0.3219571
37	$126094\,x^2 + 84621\,x - 116445$	$[0, -1, -1]$	0.1757646
38	$-242539\,x^2 + 41473\,x + 84621$	$[1, 2, 0]$	0.3027553
39	$-284012\,x^2 - 285687\,x + 327160$	$[1, -2, 0]$	0.1749961
40	$611172\,x^2 + 1675\,x - 285687$	$[1, -2, 0]$	0.2843288
41	$-896859\,x^2 + 609497\,x + 1675$	$[1, -2, -2]$	0.2850549
42	$-609497\,x^2 - 898534\,x + 896859$	$[1, -2, -1]$	0.1952281
43	$-1506356\,x^2 - 289037\,x + 898534$	$[0, 1, 1]$	0.2554551

Let $\xi = 0.5436...$ be a root of $x^3 + x^2 + x - 1$.

1	$-x^2 - x + 1$	$[1, -1, -1]$	0.1607132
2	$2x - 1$	$[1, -1, -1]$	0.3495121
3	$2x^2 - x$	$[1, -1, -1]$	0.1900259
4	$-3x^2 - 2x + 2$	$[1, -1, -1]$	0.2324587
5	$x^2 + 5x - 3$	$[1, -1, -1]$	0.3510701
6	$4x^2 - 4x + 1$	$[1, -1, -1]$	0.1221587
7	$-8x^2 - 3x + 4$	$[1, -1, -1]$	0.2656654
8	$5x^2 + 12x - 8$	$[1, -1, -1]$	0.3249885
9	$7x^2 - 13x + 5$	$[1, -1, -1]$	0.2073685
10	$-20x^2 - 2x + 7$	$[1, -1, -1]$	0.2668497
11	$18x^2 + 27x - 20$	$[1, -1, -1]$	0.2644142
12	$9x^2 - 38x + 18$	$[1, -1, -1]$	0.2847574
13	$-47x^2 + 9x + 9$	$[1, -1, -1]$	0.2368395
14	$56x^2 + 56x - 47$	$[1, -1, -1]$	0.1828037
15	$-103x + 56$	$[1, -1, -1]$	0.3362280
16	$-103x^2 + 56x$	$[1, -1, -1]$	0.1828035
17	$159x^2 + 103x - 103$	$[1, -1, -1]$	0.2368399
18	$-56x^2 - 262x + 159$	$[1, -1, -1]$	0.3496340
19	$-206x^2 + 215x - 56$	$[1, -1, -1]$	0.1280084
20	$421x^2 + 150x - 206$	$[1, -1, -1]$	0.2668557

Let $\xi = 0.7548...$ be a root of $x^3 + x^2 - 1$.

1	$-x^2+1$	$[1,0,-1]$	0.4301597
2	x^2+x-1	$[1,0,-1]$	0.3247180
3	$-x+1$	$[1,0,-1]$	0.2451223
4	$-x^2+x$	$[1,0,-1]$	0.1850374
5	$2\,x^2-1$	$[1,0,-1]$	0.5587223
6	$-2\,x^2-x+2$	$[1,0,-1]$	0.4217670
7	$x^2+2\,x-2$	$[1,0,-1]$	0.3183825
8	$x^2-2\,x+1$	$[1,0,-1]$	0.2403398
9	$-3\,x^2+x+1$	$[1,0,-1]$	0.4082111
10	$4\,x^2+x-3$	$[1,0,-1]$	0.5478213
11	$-3\,x^2-3\,x+4$	$[1,0,-1]$	0.4135381
12	$4\,x-3$	$[1,0,-1]$	0.3121706
13	$4\,x^2-3\,x$	$[1,0,-1]$	0.2356506
14	$-7\,x^2+4$	$[1,0,-1]$	0.5447802
15	$7\,x^2+4\,x-7$	$[1,0,-1]$	0.4112424
16	$-3\,x^2-7\,x+7$	$[1,0,-1]$	0.3104377
17	$-4\,x^2+7\,x-3$	$[1,0,-1]$	0.2343425
18	$11\,x^2-3\,x-4$	$[1,0,-1]$	0.4368345
19	$-14\,x^2-4\,x+11$	$[1,0,-1]$	0.5341512
20	$10\,x^2+11\,x-14$	$[1,0,-1]$	0.4032188

Let $\xi = 0.6823...$ be a root of $x^3 + x - 1$.

1	$-x+1$	$[1,-1,0]$	0.3176722
2	$-x^2+x$	$[1,-1,0]$	0.2167566
3	x^2+x-1	$[1,-1,0]$	0.1478990
4	$x^2-2\,x+1$	$[1,-1,0]$	0.4036625
5	$-2\,x^2+1$	$[1,-1,0]$	0.2754301
6	$3\,x-2$	$[1,-1,0]$	0.4228507
7	$3\,x^2-2\,x$	$[1,-1,0]$	0.2885228
8	$-2\,x^2-3\,x+3$	$[1,-1,0]$	0.1968671
9	$-3\,x^2+5\,x-2$	$[1,-1,0]$	0.3731331
10	$5\,x^2+x-3$	$[1,-1,0]$	0.2545991
11	$x^2-8\,x+5$	$[1,-1,0]$	0.4447233
12	$-8\,x^2+4\,x+1$	$[1,-1,0]$	0.3034471
13	$4\,x^2+9\,x-8$	$[1,-1,0]$	0.2620481
14	$9\,x^2-12\,x+4$	$[1,-1,0]$	0.3178715
15	$-12\,x^2-5\,x+9$	$[1,-1,0]$	0.2168926
16	$-5\,x^2+21\,x-12$	$[1,-1,0]$	0.4532250
17	$21\,x^2-7\,x-5$	$[1,-1,0]$	0.3092480
18	$-7\,x^2-26\,x+21$	$[1,-1,0]$	0.3234507
19	$-26\,x^2+28\,x-7$	$[1,-1,0]$	0.2559591
20	$28\,x^2+19\,x-26$	$[1,-1,0]$	0.1746480

Let $\xi = 0.2756...$ be a root of $x^3 + 2x^2 + 3x - 1$.

1	$-2\,x^2 - 3\,x + 1$	$[1, -3, -2]$	0.1885683
2	$x^2 + 7\,x - 2$	$[1, -3, -2]$	0.2830290
3	$5\,x^2 - 5\,x + 1$	$[1, -3, -2]$	0.03980922
4	$-15\,x^2 - 14\,x + 5$	$[1, -3, -2]$	0.09877224
5	$16\,x^2 + 50\,x - 15$	$[1, -3, -2]$	0.3025528
6	$18\,x^2 - 63\,x + 16$	$[1, -3, -2]$	0.1324192
7	$-99\,x^2 - 38\,x + 18$	$[1, -3, -2]$	0.09014652
8	$160\,x^2 + 315\,x - 99$	$[1, -3, -2]$	0.2515987
9	$-5\,x^2 - 579\,x + 160$	$[1, -3, -2]$	0.2343436
10	$-569\,x^2 + 175\,x - 5$	$[1, -3, -2]$	0.06239206
11	$1313\,x^2 + 1702\,x - 569$	$[1, -3, -2]$	0.1538979
12	$-924\,x^2 - 4508\,x + 1313$	$[1, -3, -2]$	0.2976392
13	$-2660\,x^2 + 4085\,x - 924$	$[1, -3, -2]$	0.06737753
14	$9405\,x^2 + 7056\,x - 2660$	$[1, -3, -2]$	0.09845943
15	$-11754\,x^2 - 30875\,x + 9405$	$[1, -3, -2]$	0.2925246
16	$-7367\,x^2 + 44667\,x - 11754$	$[1, -3, -2]$	0.1687838
17	$59401\,x^2 + 10347\,x - 7367$	$[1, -3, -2]$	0.08229120
18	$-108455\,x^2 - 185570\,x + 59401$	$[1, -3, -2]$	0.2214064
19	$31340\,x^2 + 384766\,x - 108455$	$[1, -3, -2]$	0.2624078
20	$322086\,x^2 - 202475\,x + 31340$	$[1, -3, -2]$	0.05069158

Let c_1 and c_2 be real numbers such that $f(x) = x^3 + c_2x^2 + c_1x - 1$ has a real root $0 < \xi < 1$. Consider the following recursion:

$$P_i(x) = -c_2P_{i-1}(x) - c_1P_{i-2}(x) + P_{i-3}(x), \quad i = 4, 5, \dots,$$

where $P_3(x) = x^2, P_2(x) = x, P_1(x) = 1$. Put $K_i = |P_i(\xi)|\overline{|P_i|}^2$.

<u>Statement</u>: Let c_1 and c_2 be nonnegative integers such that

$$c_1 \geq \begin{cases} \lfloor c_2^2/4 \rfloor + 1 & \text{if} \quad 0 \leq c_2 \leq 8 \\ \lfloor c_2^2/4 \rfloor + 2 & \text{if} \quad c_2 > 8. \end{cases}$$

Then $\sup_i K_i < 1$.

Let c_1 and c_2 be real numbers such that $f(x) = x^3 + c_2x^2 + c_1x - 1$ has a real root $0 < \xi < 1$. Consider the following recursion:

$$P_i(x) = -c_2P_{i-1}(x) - c_1P_{i-2}(x) + P_{i-3}(x), \quad i = 4, 5, \ldots,$$

where $P_3(x) = x^2, P_2(x) = x, P_1(x) = 1$. Put $K_i = |P_i(\xi)|\lceil P_i \rceil^2$.

STATEMENT: Let c_1 and c_2 be nonnegative integers such that

$$c_1 \geq \begin{cases} \lfloor c_2^2/4 \rfloor + 1 & \text{if} \quad 0 \leq c_2 \leq 8 \\ \lfloor c_2^2/4 \rfloor + 2 & \text{if} \quad c_2 > 8. \end{cases}$$

Then $\sup_i K_i < 1$.

THEOREM (Dirichlet): For any real number $\xi \notin \mathbb{A}_n$ there exist infinitely many integer polynomials $P(x)$ of degree $\leq n$ such that

$$|P(\xi)| \ll \lceil P \rceil^{-n}$$

THEOREM (Dirichlet): For any real irrational numbers $\xi_1, \ldots, \xi_n$ there exist infinitely many integer numbers $q, p_1, \ldots, p_n$ such that

$$\begin{cases} |q\xi_1 + p_1| < q^{-1-\frac{1}{n}} \\ \ldots \\ |q\xi_n + p_n| < q^{-1-\frac{1}{n}} \end{cases}$$

Let c_1 and c_2 be real numbers such that $f(x) = x^3 + c_2x^2 + c_1x - 1$ has a real root $0 < \xi < 1$. Consider the following recursions

$$P_i(x) = -c_2P_{i-1}(x) - c_1P_{i-2}(x) + P_{i-3}(x), \quad i = 4, 5, \ldots,$$

$$Q_i(x) = c_1Q_{i-1}(x) + c_2Q_{i-2}(x) + Q_{i-3}(x), \quad i = 4, 5, \ldots,$$

$$R_i(x) = c_1R_{i-1}(x) + c_2R_{i-2}(x) + R_{i-3}(x), \quad i = 4, 5, \ldots,$$

where

$$P_3(x) = x^2,\ P_2(x) = x,\ P_1(x) = 1,$$

$$Q_3(x) = c_1x - 1,\ Q_2(x) = x,\ Q_1(x) = 0,$$

$$R_3(x) = c_1x^2,\ R_2(x) = x^2,\ R_1(x) = -1.$$

Put

$$K_i = |P_i(\xi)|\overline{|P_i|}^2 \quad \text{and} \quad K_i^* = \max\left(|Q_i(\xi)|\overline{|Q_i|}^{1/2},\ |R_i(\xi)|\overline{|R_i|}^{1/2}\right)$$

STATEMENT: Let c_1 and c_2 be nonnegative integers such that

$$c_1 \geq \begin{cases} \lfloor c_2^2/4 \rfloor + 1 & \text{if} \quad 0 \leq c_2 \leq 8 \\ \lfloor c_2^2/4 \rfloor + 2 & \text{if} \quad c_2 > 8. \end{cases}$$

Then $\sup_i K_i < 1$ and $\sup_i K_i^* < 1$.

Let $\xi = 0.6823...$ be a root of $x^3 + x - 1$.

1	$-x+1$	$[1,-1,0]$	0.3176722
2	$-x^2+x$	$[1,-1,0]$	0.2167566
3	x^2+x-1	$[1,-1,0]$	0.1478990
4	$x^2-2\,x+1$	$[1,-1,0]$	0.4036625
5	$-2\,x^2+1$	$[1,-1,0]$	0.2754301
6	$3\,x-2$	$[1,-1,0]$	0.4228507
7	$3\,x^2-2\,x$	$[1,-1,0]$	0.2885228
8	$-2\,x^2-3\,x+3$	$[1,-1,0]$	0.1968671
9	$-3\,x^2+5\,x-2$	$[1,-1,0]$	0.3731331
10	$5\,x^2+x-3$	$[1,-1,0]$	0.2545991
11	$x^2-8\,x+5$	$[1,-1,0]$	0.4447233
12	$-8\,x^2+4\,x+1$	$[1,-1,0]$	0.3034471
13	$4\,x^2+9\,x-8$	$[1,-1,0]$	0.2620481
14	$9\,x^2-12\,x+4$	$[1,-1,0]$	0.3178715
15	$-12\,x^2-5\,x+9$	$[1,-1,0]$	0.2168926
16	$-5\,x^2+21\,x-12$	$[1,-1,0]$	0.4532250
17	$21\,x^2-7\,x-5$	$[1,-1,0]$	0.3092480
18	$-7\,x^2-26\,x+21$	$[1,-1,0]$	0.3234507
19	$-26\,x^2+28\,x-7$	$[1,-1,0]$	0.2559591
20	$28\,x^2+19\,x-26$	$[1,-1,0]$	0.1746480

Let $\xi = 0.6823...$ be a root of $x^3 + x - 1$.

1	$-x^2 - x + 1$	$[1, -1, -1]$	$-x + 1$	$[1, -1, 0]$
2	$-2x^2 + 1$	$[1, -1, 1]$	$-x^2 + x$	$[1, -1, 0]$
3	$-2x^2 - 3x + 3$	$[1, 3, 0]$	$x^2 + x - 1$	$[1, -1, 0]$
4	$-5x^2 - x + 3$	$[1, 2, 0]$	$x^2 - 2x + 1$	$[1, -1, 0]$
5	$-8x^2 + 4x + 1$	$[1, -2, 2]$	$-2x^2 + 1$	$[1, -1, 0]$
6	$-4x^2 - 9x + 8$	$[1, 2, -1]$	$3x - 2$	$[1, -1, 0]$
7	$-12x^2 - 5x + 9$	$[0, 1, 1]$	$3x^2 - 2x$	$[1, -1, 0]$
8	$-21x^2 + 7x + 5$	$[1, 2, 0]$	$-2x^2 - 3x + 3$	$[1, -1, 0]$
9	$-7x^2 - 26x + 21$	$[1, 2, -1]$	$-3x^2 + 5x - 2$	$[1, -1, 0]$
10	$-28x^2 - 19x + 26$	$[0, 1, 1]$	$5x^2 + x - 3$	$[1, -1, 0]$
11	$-54x^2 + 9x + 19$	$[1, 2, 0]$	$x^2 - 8x + 5$	$[1, -1, 0]$
12	$-63x^2 - 64x + 73$	$[1, 2, 0]$	$-8x^2 + 4x + 1$	$[1, -1, 0]$
13	$-136x^2 - x + 64$	$[1, 2, 0]$	$4x^2 + 9x - 8$	$[1, -1, 0]$
14	$-200x^2 + 135x + 1$	$[1, -2, 2]$	$9x^2 - 12x + 4$	$[1, -1, 0]$
15	$-135x^2 - 201x + 200$	$[1, 2, -1]$	$-12x^2 - 5x + 9$	$[1, -1, 0]$
16	$-335x^2 - 66x + 201$	$[0, 1, 1]$	$-5x^2 + 21x - 12$	$[1, -1, 0]$
17	$-536x^2 + 269x + 66$	$[1, 2, 0]$	$21x^2 - 7x - 5$	$[1, -1, 0]$
18	$-269x^2 - 602x + 536$	$[1, 2, -1]$	$-7x^2 - 26x + 21$	$[1, -1, 0]$
19	$-805x^2 - 333x + 602$	$[0, 1, 1]$	$-26x^2 + 28x - 7$	$[1, -1, 0]$
20	$-1407x^2 + 472x + 333$	$[1, 2, 0]$	$28x^2 + 19x - 26$	$[1, -1, 0]$
21	$-472x^2 - 1740x + 1407$	$[1, 2, -1]$	$19x^2 - 54x + 28$	$[1, -1, 0]$
22	$1879x^2 + 1268x - 1740$	$[0, -1, -1]$	$-54x^2 + 9x + 19$	$[1, -1, 0]$
23	$-3619x^2 + 611x + 1268$	$[1, 2, 0]$	$9x^2 + 73x - 54$	$[1, -1, 0]$
24	$-4230x^2 - 4276x + 4887$	$[1, -2, 0]$	$73x^2 - 63x + 9$	$[1, -1, 0]$
25	$9117x^2 + 46x - 4276$	$[1, -2, 0]$	$-63x^2 - 64x + 73$	$[1, -1, 0]$

Let $\xi = 0.6823...$ be a root of $x^3 + x - 1$.

1	$x^2 + x - 1$	$x - 1$
2	$2x^2 - 1$	$x^2 - x$
3	$2x^2 + 3x - 3$	$x^2 + x - 1$
4	$5x^2 + x - 3$	$x^2 - 2x + 1$
5	$8x^2 - 4x - 1$	$2x^2 - 1$
6	$4x^2 + 9x - 8$	$3x - 2$
7	$12x^2 + 5x - 9$	$3x^2 - 2x$
8	$21x^2 - 7x - 5$	$2x^2 + 3x - 3$
9	$7x^2 + 26x - 21$	$3x^2 - 5x + 2$
10	$28x^2 + 19x - 26$	$5x^2 + x - 3$
11	$54x^2 - 9x - 19$	$x^2 - 8x + 5$
12	$63x^2 + 64x - 73$	$8x^2 - 4x - 1$
13	$136x^2 + x - 64$	$4x^2 + 9x - 8$
14	$200x^2 - 135x - 1$	$9x^2 - 12x + 4$
15	$135x^2 + 201x - 200$	$12x^2 + 5x - 9$
16	$335x^2 + 66x - 201$	$5x^2 - 21x + 12$
17	$536x^2 - 269x - 66$	$21x^2 - 7x - 5$
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5	$8\,x^2 - 4\,x - 1$	$2\,x^2 - 1$
6	$4\,x^2 + 9\,x - 8$	$3\,x - 2$
7	$12\,x^2 + 5\,x - 9$	$3\,x^2 - 2\,x$
8	$21\,x^2 - 7\,x - 5$	$2\,x^2 + 3\,x - 3$
9	$7\,x^2 + 26\,x - 21$	$3\,x^2 - 5\,x + 2$
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13	$136\,x^2 + x - 64$	$4\,x^2 + 9\,x - 8$
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15	$135\,x^2 + 201\,x - 200$	$12\,x^2 + 5\,x - 9$
16	$335\,x^2 + 66\,x - 201$	$5\,x^2 - 21\,x + 12$
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20	$1407\,x^2 - 472\,x - 333$	$28\,x^2 + 19\,x - 26$
21	$472\,x^2 + 1740\,x - 1407$	$19\,x^2 - 54\,x + 28$
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23	$3619\,x^2 - 611\,x - 1268$	$9\,x^2 + 73\,x - 54$
24	$4230\,x^2 + 4276\,x - 4887$	$73\,x^2 - 63\,x + 9$
25	$9117\,x^2 + 46\,x - 4276$	$63\,x^2 + 64\,x - 73$

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Let $\xi = 0.6823...$ be a root of $x^3 + x - 1$.

1	$-x+1$	$[1,-1,0]$	0.3176722
2	$-x^2+x$	$[1,-1,0]$	0.2167566
3	x^2+x-1	$[1,-1,0]$	0.1478990
4	$x^2-2\,x+1$	$[1,-1,0]$	0.4036625
5	$-2\,x^2+1$	$[1,-1,0]$	0.2754301
6	$3\,x-2$	$[1,-1,0]$	0.4228507
7	$3\,x^2-2\,x$	$[1,-1,0]$	0.2885228
8	$-2\,x^2-3\,x+3$	$[1,-1,0]$	0.1968671
9	$-3\,x^2+5\,x-2$	$[1,-1,0]$	0.3731331
10	$5\,x^2+x-3$	$[1,-1,0]$	0.2545991
11	$x^2-8\,x+5$	$[1,-1,0]$	0.4447233
12	$-8\,x^2+4\,x+1$	$[1,-1,0]$	0.3034471
13	$4\,x^2+9\,x-8$	$[1,-1,0]$	0.2620481
14	$9\,x^2-12\,x+4$	$[1,-1,0]$	0.3178715
15	$-12\,x^2-5\,x+9$	$[1,-1,0]$	0.2168926
16	$-5\,x^2+21\,x-12$	$[1,-1,0]$	0.4532250
17	$21\,x^2-7\,x-5$	$[1,-1,0]$	0.3092480
18	$-7\,x^2-26\,x+21$	$[1,-1,0]$	0.3234507
19	$-26\,x^2+28\,x-7$	$[1,-1,0]$	0.2559591
20	$28\,x^2+19\,x-26$	$[1,-1,0]$	0.1746480

Let $\xi = 0.6180...$ be a root of $x^2 + x - 1$.

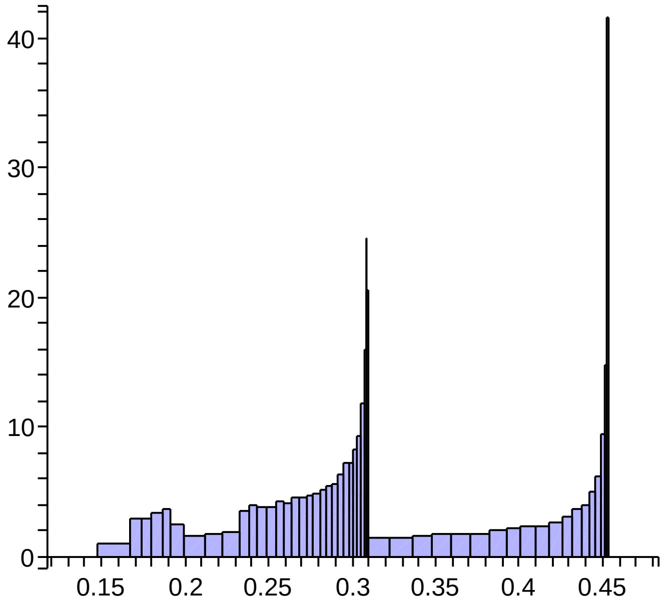
1	-1		-1
2	x		0.6180339887
3	$x-1$		0.3819660113
4	$2x-1$	$[1,1,1]$	0.4721359550
5	$3x-2$	$[1,1,1]$	0.4376941013
6	$5x-3$	$[1,1,1]$	0.4508497187
7	$8x-5$	$[1,1,1]$	0.4458247200
8	$13x-8$	$[1,1,1]$	0.4477440987
9	$21x-13$	$[1,1,1]$	0.4470109613
10	$34x-21$	$[1,1,1]$	0.4472909949
11	$55x-34$	$[1,1,1]$	0.4471840316
12	$89x-55$	$[1,1,1]$	0.4472248879
13	$144x-89$	$[1,1,1]$	0.4472092822
14	$233x-144$	$[1,1,1]$	0.4472152430
15	$377x-233$	$[1,1,1]$	0.4472129662
16	$610x-377$	$[1,1,1]$	0.4472138359
17	$987x-610$	$[1,1,1]$	0.4472135037
18	$1597x-987$	$[1,1,1]$	0.4472136306
19	$2584x-1597$	$[1,1,1]$	0.4472135821
20	$4181x-2584$	$[1,1,1]$	0.4472136006

Let $\xi = 0.6823...$ be a root of $x^3 + x - 1$.

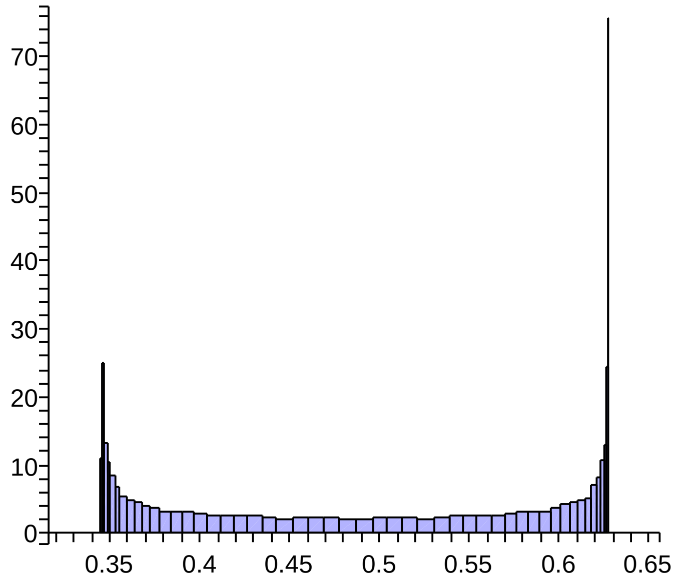
1	$-x+1$	$[1,-1,0]$	0.3176722
2	$-x^2+x$	$[1,-1,0]$	0.2167566
3	x^2+x-1	$[1,-1,0]$	0.1478990
4	$x^2-2\,x+1$	$[1,-1,0]$	0.4036625
5	$-2\,x^2+1$	$[1,-1,0]$	0.2754301
6	$3\,x-2$	$[1,-1,0]$	0.4228507
7	$3\,x^2-2\,x$	$[1,-1,0]$	0.2885228
8	$-2\,x^2-3\,x+3$	$[1,-1,0]$	0.1968671
9	$-3\,x^2+5\,x-2$	$[1,-1,0]$	0.3731331
10	$5\,x^2+x-3$	$[1,-1,0]$	0.2545991
11	$x^2-8\,x+5$	$[1,-1,0]$	0.4447233
12	$-8\,x^2+4\,x+1$	$[1,-1,0]$	0.3034471
13	$4\,x^2+9\,x-8$	$[1,-1,0]$	0.2620481
14	$9\,x^2-12\,x+4$	$[1,-1,0]$	0.3178715
15	$-12\,x^2-5\,x+9$	$[1,-1,0]$	0.2168926
16	$-5\,x^2+21\,x-12$	$[1,-1,0]$	0.4532250
17	$21\,x^2-7\,x-5$	$[1,-1,0]$	0.3092480
18	$-7\,x^2-26\,x+21$	$[1,-1,0]$	0.3234507
19	$-26\,x^2+28\,x-7$	$[1,-1,0]$	0.2559591
20	$28\,x^2+19\,x-26$	$[1,-1,0]$	0.1746480

Let $\xi = 0.6823...$ be a root of $x^3 + x - 1$.

1	$-x+1$	$[1,-1,0]$	0.3176722	0.6353444
2	$-x^2+x$	$[1,-1,0]$	0.2167566	0.4335131
3	x^2+x-1	$[1,-1,0]$	0.1478990	0.4436971
4	x^2-2x+1	$[1,-1,0]$	0.4036625	0.6054937
5	$-2x^2+1$	$[1,-1,0]$	0.2754301	0.3442877
6	$3x-2$	$[1,-1,0]$	0.4228507	0.6107843
7	$3x^2-2x$	$[1,-1,0]$	0.2885228	0.4167551
8	$-2x^2-3x+3$	$[1,-1,0]$	0.1968671	0.4812307
9	$-3x^2+5x-2$	$[1,-1,0]$	0.3731331	0.5671623
10	$5x^2+x-3$	$[1,-1,0]$	0.2545991	0.3564387
11	x^2-8x+5	$[1,-1,0]$	0.4447233	0.6253921
12	$-8x^2+4x+1$	$[1,-1,0]$	0.3034471	0.3840502
13	$4x^2+9x-8$	$[1,-1,0]$	0.2620481	0.5208611
14	$9x^2-12x+4$	$[1,-1,0]$	0.3178715	0.5319933
15	$-12x^2-5x+9$	$[1,-1,0]$	0.2168926	0.3765496
16	$-5x^2+21x-12$	$[1,-1,0]$	0.4532250	0.6269098
17	$21x^2-7x-5$	$[1,-1,0]$	0.3092480	0.3611399
18	$-7x^2-26x+21$	$[1,-1,0]$	0.3234507	0.5579046
19	$-26x^2+28x-7$	$[1,-1,0]$	0.2559591	0.4926559
20	$28x^2+19x-26$	$[1,-1,0]$	0.1746480	0.4056556



0
10
20
30
40
0.15
0.2
0.25
0.3
0.35
0.4
0.45



0
10
20
30
40
50
60
70
0.35
0.4
0.45
0.5
0.55
0.6
0.65

Definition: The beta distribution is a family of continuous probability distributions defined on the interval [0, 1] with the probability density function

$$f(x;\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} = \frac{1}{\mathrm{B}(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}$$

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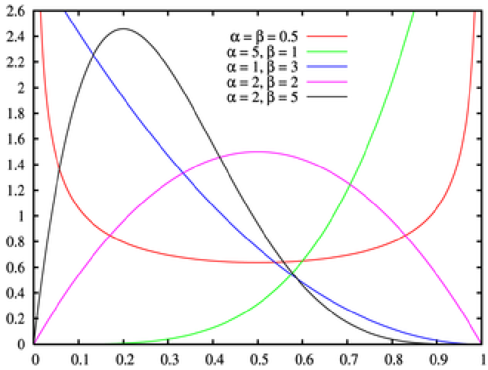

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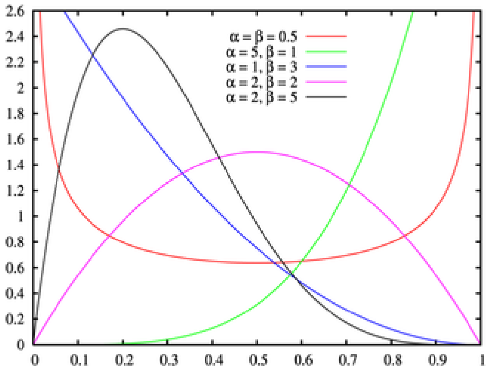

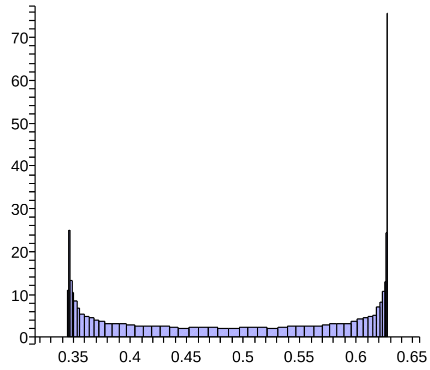

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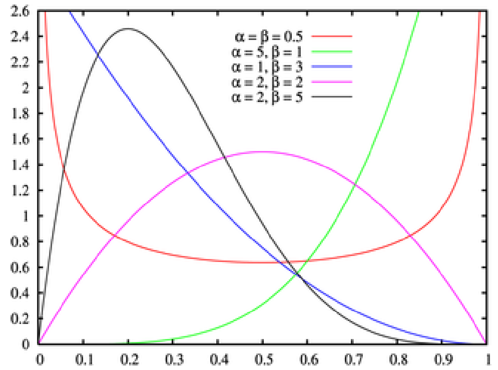

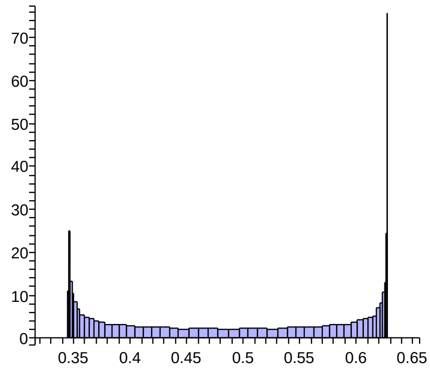

We have $\mathrm{B}(\alpha,\beta)=\int\limits_0^1 u^{\alpha-1}(1-u)^{\beta-1}du.$

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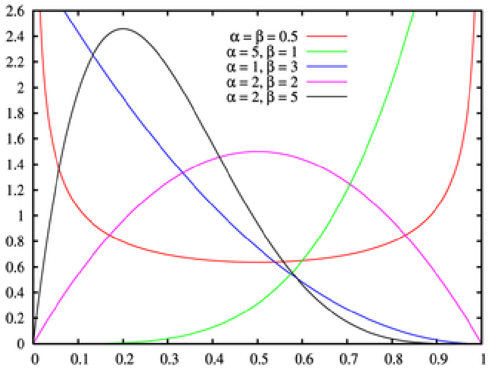
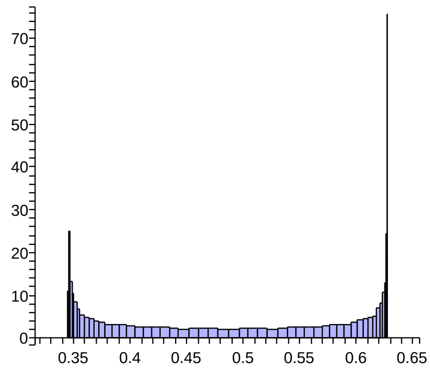
$$f(x;\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1} = \frac{1}{\mathrm{B}(\alpha,\beta)}x^{\alpha-1}(1-x)^{\beta-1}$$





We have $\mathrm{B}(\alpha,\beta) = \int\limits_0^1 u^{\alpha-1}(1-u)^{\beta-1}du$. If $\alpha=\beta=1/2$, then

$$\mathrm{B}(1/2,1/2) = \int_0^1 u^{-1/2}(1-u)^{-1/2}du = \pi$$

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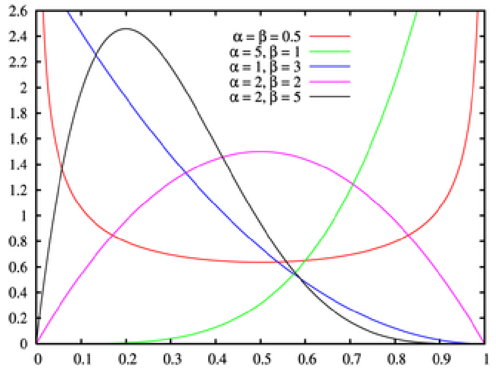

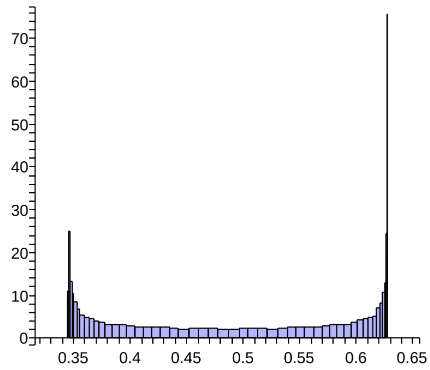

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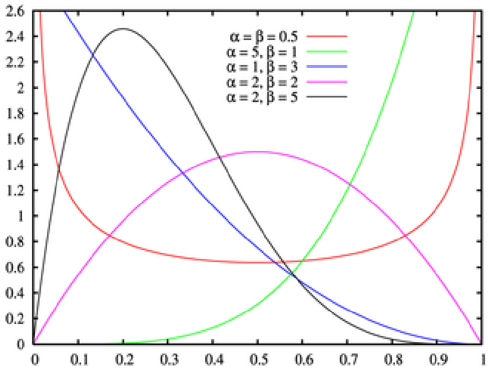

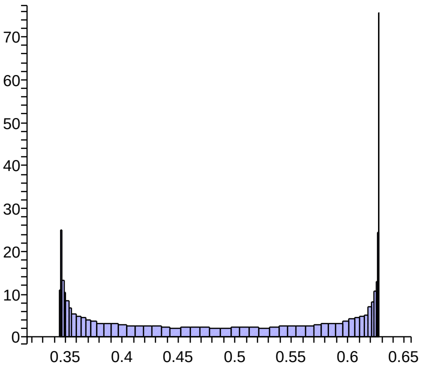

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Note that $(\arcsin\sqrt{x})' = \dfrac{1}{2\sqrt{x(1-x)}}$.

Let $\xi = 0.6823...$ be a root of $x^3 + x - 1$.

1	$-x+1$	$[1,-1,0]$	0.3176722	0.6353444
2	$-x^2+x$	$[1,-1,0]$	0.2167566	0.4335131
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$$P_{-2}(x) = 1$$

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$$P_i(x) = xP_{i-1}(x) \bmod x^3+x-1$$

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$$P_i(x) = P_{i-3}(x) - P_{i-2}(x)$$

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$$P_i(x) = x^{i+2} \bmod x^3+x-1$$

Let c_1 and c_2 be real numbers such that $f(x) = x^3 + c_2x^2 + c_1x - 1$ has a real root $0 < \xi < 1$. Consider the following recursions

$$P_i(x) = -c_2P_{i-1}(x) - c_1P_{i-2}(x) + P_{i-3}(x), \quad i = 4, 5, \dots,$$

$$Q_i(x) = c_1Q_{i-1}(x) + c_2Q_{i-2}(x) + Q_{i-3}(x), \quad i = 4, 5, \dots,$$

$$R_i(x) = c_1R_{i-1}(x) + c_2R_{i-2}(x) + R_{i-3}(x), \quad i = 4, 5, \dots,$$

where

$$P_3(x) = x^2,\ P_2(x) = x,\ P_1(x) = 1,$$

$$Q_3(x) = c_1x - 1,\ Q_2(x) = x,\ Q_1(x) = 0,$$

$$R_3(x) = c_1x^2,\ R_2(x) = x^2,\ R_1(x) = -1.$$

Put

$$K_i = |P_i(\xi)|\lceil P_i \rceil^2 \quad \text{and} \quad K_i^* = \max\left(|Q_i(\xi)|\lceil Q_i \rceil^{1/2},\ |R_i(\xi)|\lceil R_i \rceil^{1/2}\right)$$

STATEMENT: Let c_1 and c_2 be nonnegative integers such that

$$c_1 \geq \begin{cases} \lfloor c_2^2/4 \rfloor + 1 & \text{if} \quad 0 \leq c_2 \leq 8 \\ \lfloor c_2^2/4 \rfloor + 2 & \text{if} \quad c_2 > 8. \end{cases}$$

Then $\sup_i K_i < 1$ and $\sup_i K_i^* < 1$.

Let c_1 and c_2 be real numbers such that $f(x) = x^3 + c_2x^2 + c_1x - 1$ has a real root $0 < \xi < 1$. Put

$$K_i = |P_i(\xi)|\,\overline{|P_i|}^2 \quad \text{and} \quad K_i^* = \max\left(|Q_i(\xi)||Q_i|^{1/2},\ |R_i(\xi)||R_i|^{1/2}\right)$$

where

$$P_i(x) = x^i \mod f(x)$$

and

$$\begin{bmatrix} Q_i(x) \\ R_i(x) \end{bmatrix} = (-1)^i \begin{bmatrix} c_2x & x \\ c_2x^2 - 1 & x^2 \end{bmatrix}^i \begin{bmatrix} 0 \\ -1 \end{bmatrix} \mod f(x)$$

<u>Statement</u>: Let c_1 and c_2 be nonnegative integers such that

$$c_1 \geq \begin{cases} \lfloor c_2^2/4 \rfloor + 1 & \text{if} \quad 0 \leq c_2 \leq 8 \\ \lfloor c_2^2/4 \rfloor + 2 & \text{if} \quad c_2 > 8 \end{cases}$$

Then

$$\sup_i K_i < 1 \quad \text{and} \quad \sup_i K_i^* < 1$$

Let $\xi = 0.6823...$ be a root of $x^3 + x - 1$.

1	$-x+1$	$[1,-1,0]$	0.3176722	0.6353444
2	$-x^2+x$	$[1,-1,0]$	0.2167566	0.4335131
3	x^2+x-1	$[1,-1,0]$	0.1478990	0.4436971
4	x^2-2x+1	$[1,-1,0]$	0.4036625	0.6054937
5	$-2x^2+1$	$[1,-1,0]$	0.2754301	0.3442877
6	$3x-2$	$[1,-1,0]$	0.4228507	0.6107843
7	$3x^2-2x$	$[1,-1,0]$	0.2885228	0.4167551
8	$-2x^2-3x+3$	$[1,-1,0]$	0.1968671	0.4812307
9	$-3x^2+5x-2$	$[1,-1,0]$	0.3731331	0.5671623
10	$5x^2+x-3$	$[1,-1,0]$	0.2545991	0.3564387
11	x^2-8x+5	$[1,-1,0]$	0.4447233	0.6253921
12	$-8x^2+4x+1$	$[1,-1,0]$	0.3034471	0.3840502
13	$4x^2+9x-8$	$[1,-1,0]$	0.2620481	0.5208611
14	$9x^2-12x+4$	$[1,-1,0]$	0.3178715	0.5319933
15	$-12x^2-5x+9$	$[1,-1,0]$	0.2168926	0.3765496
16	$-5x^2+21x-12$	$[1,-1,0]$	0.4532250	0.6269098
17	$21x^2-7x-5$	$[1,-1,0]$	0.3092480	0.3611399
18	$-7x^2-26x+21$	$[1,-1,0]$	0.3234507	0.5579046
19	$-26x^2+28x-7$	$[1,-1,0]$	0.2559591	0.4926559
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$$P_i(x) = x^{i+2} \bmod x^3+x-1$$

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$$\begin{aligned} P_n(\xi) &= \xi^n = a_n + b_n\xi + c_n\xi^2, \\ P_n(\xi_1) &= \xi_1^n = a_n + b_n\xi_1 + c_n\xi_1^2, \\ P_n(\xi_2) &= \xi_2^n = a_n + b_n\xi_2 + c_n\xi_2^2. \end{aligned} \tag{1}$$

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In other words,

$$\begin{pmatrix} \xi^n \\ \xi_1^n \\ \xi_2^n \end{pmatrix} = V \begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix}, \quad \text{where} \quad V = \begin{pmatrix} 1 & \xi & \xi^2 \\ 1 & \xi_1 & \xi_1^2 \\ 1 & \xi_2 & \xi_2^2 \end{pmatrix}$$

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is the Vandermonde matrix. From this it follows that

$$\begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} = V^{-1} \begin{pmatrix} \xi^n \\ \xi_1^n \\ \xi_2^n \end{pmatrix}.$$

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$$|\xi|^{-1/2} = |\xi_1| = |\xi_2|, \tag{2}$$

because ξ_1 and ξ_2 are complex conjugates.

Let $f(x) = x^3 + x - 1$ and let $\xi \in \mathbb{R}, \xi_1 = \overline{\xi_2} \in \mathbb{C}$ be roots of this polynomial. Put $P_i(x) = x^{i+2} \bmod f(x)$. Since $f(\xi) = f(\xi_1) = f(\xi_2) = 0$, we have

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$$|\xi|^{-1/2} = |\xi_1| = |\xi_2|, \tag{2}$$

because ξ_1 and ξ_2 are complex conjugates. Applying (2) to (1), we get

$$\begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} = V^{-1}\xi^{-n/2} \begin{pmatrix} \xi^{3n/2} \\ \cos(n\beta) + i\sin(n\beta) \\ \cos(n\beta) - i\sin(n\beta) \end{pmatrix},$$

where $\beta = \arg(\xi_1)$.

Let $f(x) = x^3 + x - 1$ and let $\xi \in \mathbb{R}, \xi_1 = \overline{\xi_2} \in \mathbb{C}$ be roots of this polynomial. Put $P_i(x) = x^{i+2} \bmod f(x)$. Since $f(\xi) = f(\xi_1) = f(\xi_2) = 0$, we have

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where $\beta = \arg(\xi_1)$.

Let $f(x)=x^3+x-1$ and let $\xi\in\mathbb{R},\xi_1=\overline{\xi_2}\in\mathbb{C}$ be roots of this polynomial. Put $P_i(x)=x^{i+2}\bmod f(x)$. Since $f(\xi)=f(\xi_1)=f(\xi_2)=0$, we have

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where $\beta=\arg(\xi_1)$. Observe that for large n, $\xi^{3n/2}$ is much smaller than $|\cos(n\beta)\pm i\sin(n\beta)|$.

Let $f(x) = x^3 + x - 1$ and let $\xi \in \mathbb{R}, \xi_1 = \overline{\xi_2} \in \mathbb{C}$ be roots of this polynomial. Put $P_i(x) = x^{i+2} \bmod f(x)$. Since $f(\xi) = f(\xi_1) = f(\xi_2) = 0$, we have

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where $\beta = \arg(\xi_1)$. Observe that for large n, $\xi^{3n/2}$ is much smaller than $|\cos(n\beta) \pm i\sin(n\beta)|$.

Let $f(x) = x^3 + x - 1$ and let $\xi \in \mathbb{R}, \xi_1 = \overline{\xi_2} \in \mathbb{C}$ be roots of this polynomial. Put $P_i(x) = x^{i+2} \bmod f(x)$. Since $f(\xi) = f(\xi_1) = f(\xi_2) = 0$, we have

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where $\beta = \arg(\xi_1)$. Observe that for large n, $\xi^{3n/2}$ is much smaller than $|\cos(n\beta) \pm i\sin(n\beta)|$. This can be rewritten as

$$\xi^{n/2} \begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} \approx \frac{2i}{D} \begin{pmatrix} \xi^{3/2}\sin((n-1)\beta) - \sin((n-2)\beta) \\ \xi^{-1}\sin((n-2)\beta) - \xi^2\sin(n\beta) \\ \xi\sin(n\beta) - \xi^{-1/2}\sin((n-1)\beta) \end{pmatrix},$$

where D is the determinant of V.

Let $f(x) = x^3 + x - 1$ and let $\xi \in \mathbb{R}, \xi_1 = \overline{\xi_2} \in \mathbb{C}$ be roots of this polynomial. Put $P_i(x) = x^{i+2} \bmod f(x)$. Since $f(\xi) = f(\xi_1) = f(\xi_2) = 0$, we have

$$\begin{matrix} P_n(\xi) = \xi^n = a_n + b_n\xi + c_n\xi^2 \\ P_n(\xi_1) = \xi_1^n = a_n + b_n\xi_1 + c_n\xi_1^2 \\ P_n(\xi_2) = \xi_2^n = a_n + b_n\xi_2 + c_n\xi_2^2 \end{matrix} \Leftrightarrow \xi^{n/2} \begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} \approx \frac{2i}{D} \begin{pmatrix} \xi^{3/2}\sin((n-1)\beta) - \sin((n-2)\beta) \\ \xi^{-1}\sin((n-2)\beta) - \xi^2\sin(n\beta) \\ \xi\sin(n\beta) - \xi^{-1/2}\sin((n-1)\beta) \end{pmatrix},$$

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where D is the determinant of V. Put

$$g_1(n) = \xi^{3/2}\sin((n-1)\beta) - \sin((n-2)\beta)$$

$$g_2(n) = \xi^{-1}\sin((n-2)\beta) - \xi^2\sin(n\beta)$$

$$g_3(n) = \xi\sin(n\beta) - \xi^{-1/2}\sin((n-1)\beta)$$

Let $f(x) = x^3 + x - 1$ and let $\xi \in \mathbb{R}, \xi_1 = \overline{\xi_2} \in \mathbb{C}$ be roots of this polynomial. Put $P_i(x) = x^{i+2} \bmod f(x)$. Since $f(\xi) = f(\xi_1) = f(\xi_2) = 0$, we have

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where D is the determinant of V. Put

$$\begin{aligned} g_1(n) &= \xi^{3/2}\sin((n-1)\beta) - \sin((n-2)\beta) \\ g_2(n) &= \xi^{-1}\sin((n-2)\beta) - \xi^2\sin(n\beta) \\ g_3(n) &= \xi\sin(n\beta) - \xi^{-1/2}\sin((n-1)\beta) \end{aligned} \quad \Rightarrow \quad \begin{aligned} (P_n(\xi))^{1/2}a_n &\approx \frac{2i}{D}g_1(n) \\ (P_n(\xi))^{1/2}b_n &\approx \frac{2i}{D}g_2(n) \\ (P_n(\xi))^{1/2}c_n &\approx \frac{2i}{D}g_3(n) \end{aligned}$$

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where D is the determinant of V. Put

$$\begin{aligned} g_1(n) &= \xi^{3/2}\sin((n-1)\beta) - \sin((n-2)\beta) \\ g_2(n) &= \xi^{-1}\sin((n-2)\beta) - \xi^2\sin(n\beta) \\ g_3(n) &= \xi\sin(n\beta) - \xi^{-1/2}\sin((n-1)\beta) \end{aligned} \quad \Rightarrow \quad \begin{aligned} (P_n(\xi))^{1/2}a_n &\approx \frac{2i}{D}g_1(n) \\ (P_n(\xi))^{1/2}b_n &\approx \frac{2i}{D}g_2(n) \\ (P_n(\xi))^{1/2}c_n &\approx \frac{2i}{D}g_3(n) \end{aligned}$$

We show that the points $(g_1(n), g_2(n), g_3(n))$ trace an ellipse in 3-dimensional space.

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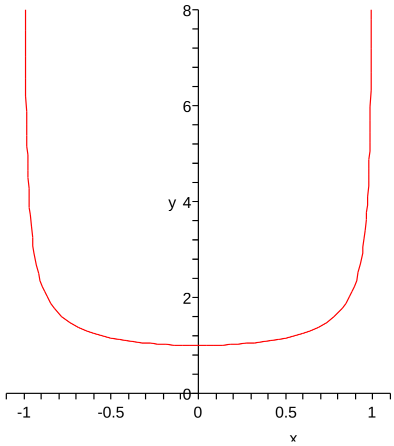

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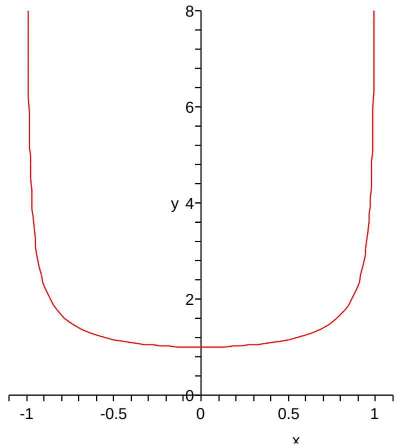

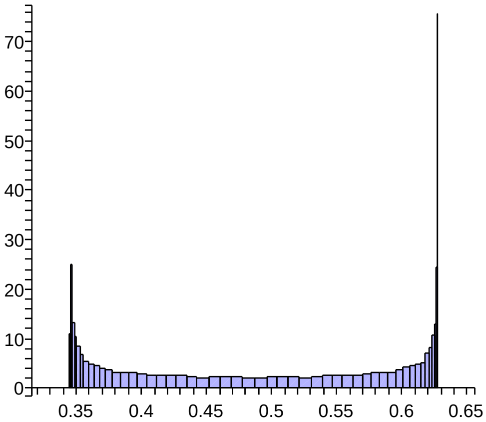

All integer irreducible polynomials of height 1 with a real root $0 < \xi < 1$.

1	$x^3 + x - 1$	0.6823278038	x^n
2	$x^3 + x^2 - 1$	0.7548776662	x^n
3	$x^3 + x^2 + x - 1$	0.5436890127	x^n

Integer irreducible polynomials of height 2 with a real root $0 < \xi < 1$.

1*	$x^3 - 2x^2 - x + 1$	$-.8019377358, .5549581321, 2.246979604$	$x^n(x-1)^m$
2*	$x^3 - x^2 - 2x + 1$	$-1.246979604, 0.4450418679, 1.801937736$	$x^n(x-1)^m$
3	$x^3 - x^2 + 2x - 1$	0.5698402910	x^n
4	$x^3 + 2x - 1$	0.4533976515	x^n
5	$x^3 + x^2 + x - 2$	0.8105357138	$(x-1)^n$
6	$x^3 + x^2 + 2x - 2$	0.6506291914	$(x^2+x-1)^n$
7	$x^3 + x^2 + 2x - 1$	0.3926467817	x^n
8*	$x^3 + 2x^2 - x - 1$	$-2.246979604, -0.5549581321, 0.8019377358$	$x^n(x^2+2x-2)^m$
9	$x^3 + 2x^2 + x - 2$	0.6956207696	$(x^2+x-1)^n$
10	$x^3 + 2x^2 + x - 1$	0.4655712319	x^n
11	$x^3 + 2x^2 + 2x - 1$	0.3532099642	x^n
12*	$2x^3 - 2x^2 - 2x + 1$	$-0.8546376797, 0.4030317168, 1.451605963$	$x^n(x-1)^m$
13	$2x^3 - 2x^2 + 2x - 1$	0.6477988713	x^n
14	$2x^3 - x^2 + x - 1$	0.7389836215	$(2x^2-x)^n$
15	$2x^3 - x^2 + 2x - 2$	0.8037608834	$(4x^2-2x-1)^n$
16	$2x^3 - 1$	0.7937005260	$(2x^2-1)^n$
17	$2x^3 + x - 2$	0.8351223485	$(6x-5)^n$
18	$2x^3 + x - 1$	0.5897545123	$(2x-1)^n$
19	$2x^3 + 2x - 1$	0.4238537991	x^n
20	$2x^3 + x^2 - x - 1$	0.8294835410	$(2x^2+x-2)^n$
21	$2x^3 + x^2 - 2$	0.8580943295	$(6x^2+3x-7)^n$
22	$2x^3 + x^2 - 1$	0.6572981061	$(2x^2-x)^n$
23	$2x^3 + x^2 + x - 2$	0.7227141772	$(20x^2-2x-9)^n$
24	$2x^3 + x^2 + 2x - 2$	0.6014906292	$(10x^2-11x+3)^n$
25	$2x^3 + x^2 + 2x - 1$	0.3760858894	$(2x^2-x)^n$
26*	$2x^3 + 2x^2 - 2x - 1$	$-1.451605963, -0.4030317168, 0.8546376797$	$(x-1)^n(2x^2+2x-3)^m$
27	$2x^3 + 2x^2 - x - 2$	0.8755503512	$(4x^2-3)^n$
28	$2x^3 + 2x^2 - 1$	0.5651977174	$(2x^2-1)^n$
29	$2x^3 + 2x^2 + x - 2$	0.6427366759	$(14x-9)^n$
30	$2x^3 + 2x^2 + x - 1$	0.4406197005	$(2x-1)^n$
31	$2x^3 + 2x^2 + 2x - 1$	0.3425080314	x^n

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Let $\mathbb{Q}(\alpha)$ be a number field with r_1 real embeddings and $2r_2$ imaginary embeddings and let $r = r_1 + r_2 - 1$. Then the multiplicative group of units U of $\mathbb{Q}(\alpha)$ has the form

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DEFINITION′: A number field is the totality of all expressions that can be constructed from an algebraic number α of degree n by repeated additions, subtractions, multiplications, and divisions.

DEFINITION: A unit is an element in a ring that has a multiplicative inverse.

Let $\mathbb{Q}(\alpha)$ be a number field with r_1 real embeddings and $2r_2$ imaginary embeddings and let $r = r_1 + r_2 - 1$. Then the multiplicative group of units U of $\mathbb{Q}(\alpha)$ has the form

$$U = \{\zeta^{e_0}\epsilon_1^{e_1}\epsilon_2^{e_2}\ldots\epsilon_r^{e_r} : e_i \in \mathbb{Z}\},$$

where ζ is a primitive root of unity and ϵ_i are called the fundamental units of $\mathbb{Q}(\alpha)$.

EXAMPLE: For $\mathbb{Q}(\sqrt{2})$ the fundamental unit is $1 + \sqrt{2}$. For $\mathbb{Q}((\sqrt{5} - 1)/2)$ the fundamental unit is $(\sqrt{5} - 1)/2$.

α	f. unit
$x^2 - 2$	$x + 1$
$x^2 + x - 1$	x
$x^3 + x - 1$	x

All integer irreducible polynomials of height 1 with a real root $0 < \xi < 1$.

1	$x^3 + x - 1$	0.6823278038	x^n
2	$x^3 + x^2 - 1$	0.7548776662	x^n
3	$x^3 + x^2 + x - 1$	0.5436890127	x^n

Integer irreducible polynomials of height 2 with a real root $0 < \xi < 1$.

1*	$x^3 - 2x^2 - x + 1$	$-.8019377358, .5549581321, 2.246979604$	$x^n(x-1)^m$
2*	$x^3 - x^2 - 2x + 1$	$-1.246979604, 0.4450418679, 1.801937736$	$x^n(x-1)^m$
3	$x^3 - x^2 + 2x - 1$	0.5698402910	x^n
4	$x^3 + 2x - 1$	0.4533976515	x^n
5	$x^3 + x^2 + x - 2$	0.8105357138	$(x-1)^n$
6	$x^3 + x^2 + 2x - 2$	0.6506291914	$(x^2+x-1)^n$
7	$x^3 + x^2 + 2x - 1$	0.3926467817	x^n
8*	$x^3 + 2x^2 - x - 1$	$-2.246979604, -0.5549581321, 0.8019377358$	$x^n(x^2+2x-2)^m$
9	$x^3 + 2x^2 + x - 2$	0.6956207696	$(x^2+x-1)^n$
10	$x^3 + 2x^2 + x - 1$	0.4655712319	x^n
11	$x^3 + 2x^2 + 2x - 1$	0.3532099642	x^n
12*	$2x^3 - 2x^2 - 2x + 1$	$-0.8546376797, 0.4030317168, 1.451605963$	$x^n(x-1)^m$
13	$2x^3 - 2x^2 + 2x - 1$	0.6477988713	x^n
14	$2x^3 - x^2 + x - 1$	0.7389836215	$(2x^2-x)^n$
15	$2x^3 - x^2 + 2x - 2$	0.8037608834	$(4x^2-2x-1)^n$
16	$2x^3 - 1$	0.7937005260	$(2x^2-1)^n$
17	$2x^3 + x - 2$	0.8351223485	$(6x-5)^n$
18	$2x^3 + x - 1$	0.5897545123	$(2x-1)^n$
19	$2x^3 + 2x - 1$	0.4238537991	x^n
20	$2x^3 + x^2 - x - 1$	0.8294835410	$(2x^2+x-2)^n$
21	$2x^3 + x^2 - 2$	0.8580943295	$(6x^2+3x-7)^n$
22	$2x^3 + x^2 - 1$	0.6572981061	$(2x^2-x)^n$
23	$2x^3 + x^2 + x - 2$	0.7227141772	$(20x^2-2x-9)^n$
24	$2x^3 + x^2 + 2x - 2$	0.6014906292	$(10x^2-11x+3)^n$
25	$2x^3 + x^2 + 2x - 1$	0.3760858894	$(2x^2-x)^n$
26*	$2x^3 + 2x^2 - 2x - 1$	$-1.451605963, -0.4030317168, 0.8546376797$	$(x-1)^n(2x^2+2x-3)^m$
27	$2x^3 + 2x^2 - x - 2$	0.8755503512	$(4x^2-3)^n$
28	$2x^3 + 2x^2 - 1$	0.5651977174	$(2x^2-1)^n$
29	$2x^3 + 2x^2 + x - 2$	0.6427366759	$(14x-9)^n$
30	$2x^3 + 2x^2 + x - 1$	0.4406197005	$(2x-1)^n$
31	$2x^3 + 2x^2 + 2x - 1$	0.3425080314	x^n