On approximation of real numbers by algebraic numbers of bounded degree

by

K. I. Tsishchanka

THEOREM 1 **(Dirichlet, 1842): For any real irrational number ξ there exist infinitely many rational numbers p/q such that**

$$\left|\xi - \frac{p}{q}\right| < \frac{1}{q^2}$$

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Theorem 1′: **For any real irrational number ξ there exist infinitely many polynomials $P(x) = ax + b$ with integer coefficients such that**

$$|P(\xi)| < c(\xi)\lceil P \rceil^{-1}, \quad \lceil P \rceil = \max\{|a|, |b|\}.$$

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<u>Theorem 2</u>: For any real number $\xi \notin A_n$ there exist infinitely many polynomials $P(x) \in Z[x]$ of degree $\leq n$ such that

$$|P(\xi)| < c(\xi, n)\lceil P \rceil^{-n}.$$

$$
\begin{array}{ccc}
\left| \boldsymbol{\xi} - \dfrac{\boldsymbol{p}}{\boldsymbol{q}} \right| < \boldsymbol{q}^{-2} & \longrightarrow & |\boldsymbol{q\xi} - \boldsymbol{p}| < \boldsymbol{q}^{-1} \\
 & & \downarrow \\
|\boldsymbol{\xi} - \boldsymbol{\alpha}| \ll \boldsymbol{H}(\boldsymbol{\alpha})^{-n-1} & \longleftarrow & |\boldsymbol{P}(\boldsymbol{\xi})| \ll \lceil \boldsymbol{P} \rceil^{-n}
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$n = 1 \;\Rightarrow\; |\xi - \alpha| \ll H(\alpha)^{-2}$ (Dirichlet,1842)

$n = 2 \;\Rightarrow\; |\xi - \alpha| \ll H(\alpha)^{-3}$ (Davenport – Schmidt, 1967)

$n \geq 3 \;\Rightarrow\; |\xi - \alpha| \ll H(\alpha)^{-\frac{n}{2}-\frac{3}{2}}$ (Wirsing, 1961)

Theorem 5 (Wirsing, 1961): **For any real number $\xi \notin A_n$ there exist infinitely many algebraic numbers $\alpha \in A_n$ with**

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LEMMA 1: **We have**

$$|\xi - \alpha| \ll \frac{|P(\xi)|}{|P'(\xi)|}$$

where α is the root of P closest to ξ.

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<u>Lemma 1</u>: **We have**

$$|\xi - \alpha| \ll \frac{|P(\xi)|}{|P'(\xi)|}$$

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<u>Lemma 2</u>: **There are infinitely many polynomials P, $Q \in Z[x]$ of degree $\leq n$, such that**

$$|P(\xi)| \ll \overline{|P|}^{-n}$$
$$|Q(\xi)| \ll \overline{|P|}^{-n}$$
$$\overline{|Q|} \ll \overline{|P|}$$

and

P, Q have no common root

<u>Lemma 3</u>: **Let $P, Q \in Z[x]$ be polynomials of degree d with $1 < d \leq n$. Suppose that P and Q have no common root. Then at least one of the following estimates is true:**

$$1 \ll \max\{|P(\xi)|, |Q(\xi)|\}^2 \max\left\{\overline{P}, \overline{Q}\right\}^{2n-2}$$

$$1 \ll \max\left\{|P(\xi)||P'(\xi)||Q'(\xi)|, |Q(\xi)||P'(\xi)|^2\right\} \overline{P}^{n-2}\overline{Q}^{n-1}$$

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LEMMA 3: **Let $P, Q \in Z[x]$ be polynomials of degree d with $1 < d \leq n$. Suppose that P and Q have no common root. Then at least one of the following estimates is true:**

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$$|P(\xi)| \ll \lceil P \rceil^{-n} \quad |Q(\xi)| \ll \lceil P \rceil^{-n} \quad \lceil Q \rceil \ll \lceil P \rceil$$

$$|P'(\xi)| \ll |P(\xi)| \lceil P \rceil^{\omega} \quad |Q'(\xi)| \ll |Q(\xi)| \lceil Q \rceil^{\omega}$$

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Theorem 6 (Wirsing, 1961): For any real number $\xi \notin A_n$ there exist infinitely many algebraic numbers $\alpha \in A_n$ with

$$|\xi - \alpha| \ll H(\alpha)^{-\frac{n}{2}-\lambda_n}, \quad \lim_{n\to\infty} \lambda_n = 2$$

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n	Th. 5, 1961	Th. 6, 1961	Th. 7, 1993	Th. 8, 2005	Conjecture
3	3	3.28	3.5	3.73	4
4	3.5	3.82	4.12	4.45	5
5	4	4.35	4.71	5.14	6
6	4.5	4.87	5.28	5.76	7
7	5	5.39	5.84	6.36	8
8	5.5	5.9	6.39	6.93	9
9	6	6.41	6.93	7.50	10
10	6.5	6.92	7.47	8.06	11
15	9	9.44	10.09	10.77	16
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$$\begin{cases} |\xi_1 - \alpha_1| \ll \lceil P \rceil^{-\frac{n}{4}-\frac{3}{4}} \\ \\ |\xi_2 - \alpha_2| \ll \lceil P \rceil^{-\frac{n}{4}-\frac{3}{4}} \end{cases}$$

where $P(x) \in Z[x]$, $P(\alpha_1) = P(\alpha_2) = 0$, $\deg P \le n$.

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THEOREM 5 (**Wirsing, 1961**): **For any real number** $\xi \notin A_n$ **there exist infinitely many algebraic numbers** $\alpha \in A_n$ **with**

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where $P(x) \in Z[x],\ \ P(\alpha_1) = P(\alpha_2) = 0$, $\deg P \le n$.

LEMMA 2: **There are infinitely many polynomials** $P,\ Q \in Z[x]$ **of degree** $\le n$, **such that**

$$\boxed{\begin{array}{l} |P(\xi)| \ll \overline{|P|}^{-n} \\ |Q(\xi)| \ll \overline{|P|}^{-n} \\ \overline{|Q|} \ll \overline{|P|} \end{array}} \quad \text{and} \quad \boxed{P, Q \text{ have no common root}}$$

<u>LEMMA 2</u>: **There are infinitely many polynomials P, $Q \in Z[x]$ of degree $\leq n$, such that**

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THEOREM 2: **For any real number $\xi \notin A_n$ there exist infinitely many polynomials $P(x) \in Z[x]$ of degree $\leq n$ such that**

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THEOREM 2′: **For any real number $\xi \notin A_n$ there exist infinitely many irreducible polynomials $P(x) \in Z[x]$ of degree $\leq n$ such that**

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Lemma 2: **There are infinitely many polynomials P, $Q \in Z[x]$ of degree $\leq n$, such that**

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Theorem: **For any real numbers $\xi_1, \xi_2 \notin A_n$ there exist infinitely many polynomials $P(x) \in Z[x]$ of degree $\leq n$ such that**

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COUNTER-EXAMPLE (Roy-Waldschmidt, 2001): For any sufficiently large n there exist real numbers ξ_1 and ξ_2 such that

$$\max\{|\xi_1 - \alpha_1|, |\xi_2 - \alpha_2|\} > \lceil P \rceil^{-\frac{3}{2}\sqrt{3n}}$$

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$$\xi_1 = \sum_{i=0}^{\infty} 2^{-\lfloor 3n^{i+1/2} \rfloor} \quad \text{and} \quad \xi_2 = \sum_{i=0}^{\infty} 2^{-\lfloor 3n^{i+1} \rfloor}$$

THEOREM 9 **(Davenport - Schmidt, 1968): Let $n \geq 3$. Let ξ be real, but not algebraic of degree ≤ 2. Then there are infinitely many algebraic integers α of degree ≤ 3 which satisfy**

$$0 < |\xi - \alpha| \ll H(\alpha)^{-\eta},$$

where $\eta = \frac{1}{2}(3 + \sqrt{5}) = 2.618...$

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CONJECTURE: **Let ξ be real, but is not algebraic of degree $\leq n$. Suppose $\epsilon > 0$. Then there are infinitely many real algebraic integers α of degree $\leq n$ with**

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Counter-example (Roy, 2001): There exist real numbers ξ such that for any algebraic integer α of degree ≤ 3, we have

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4

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$$\boldsymbol{\xi} = \cfrac{1}{3 + \cfrac{1}{2+\cfrac{1}{4+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{4+\cfrac{1}{2+\cfrac{1}{4+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{4+}}}}}}}}}}}$$

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$$(\sqrt{2})^2 = 2$$

$$(\sqrt{2})^2 - 1 = 1$$

$$(\sqrt{2} - 1)(\sqrt{2} + 1) = 1$$

$$\sqrt{2} - 1 = \frac{1}{1 + \sqrt{2}}$$

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$$\sqrt{2} = 1 + \frac{1}{1 + \sqrt{2}}$$

$$\sqrt{2} = 1 + \frac{1}{1 + \sqrt{2}}$$

$$\sqrt{2} = 1 + \cfrac{1}{1 + 1 + \cfrac{1}{1 + \sqrt{2}}}$$

$$\sqrt{2} = 1 + \frac{1}{1 + \sqrt{2}}$$

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{1 + \sqrt{2}}} \qquad \sqrt{2} = 1 + \frac{1}{1 + \sqrt{2}}$$

$$\sqrt{\mathbf{2}} = \mathbf{1} + \cfrac{\mathbf{1}}{\mathbf{2} + \cfrac{\mathbf{1}}{\mathbf{2} + \cfrac{\mathbf{1}}{\mathbf{1} + \sqrt{\mathbf{2}}}}}$$

$$\sqrt{\mathbf{2}} = \mathbf{1} + \frac{\mathbf{1}}{\mathbf{1} + \sqrt{\mathbf{2}}}$$

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{1 + \sqrt{2}}}}} \qquad \sqrt{2} = 1 + \frac{1}{1 + \sqrt{2}}$$

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{1 + \sqrt{2}}}}}}$$

$$\sqrt{2} = 1 + \frac{1}{1 + \sqrt{2}}$$

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{1 + \sqrt{2}}}}}}} \qquad \sqrt{2} = 1 + \frac{1}{1 + \sqrt{2}}$$

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{1 + \sqrt{2}}}}}}}}$$

$$\sqrt{2} = 1 + \frac{1}{1 + \sqrt{2}}$$

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{1 + \sqrt{2}}}}}}}}}$$

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{1 + \sqrt{2}}}}}}}}}}$$

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{1 + \sqrt{2}}}}}}}}}}}$$

$$\mathbf{4+\sqrt[7]{-1+\sqrt[23]{\frac{2\sqrt[3]{-37+\sqrt{6384+e^{\sin 47^\circ}}}+87}{76549-\cot 19^\circ}}}}$$

$$\sqrt{2}$$

$$\boldsymbol{\sqrt{2} = 1 + 0.4142135...}$$

$$\sqrt{\mathbf{2}} = \mathbf{1} + \frac{\mathbf{1}}{\mathbf{2.4142135...}}$$

$$\sqrt{2} = 1 + \frac{1}{2 + 0.4142135...}$$

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2.4142135\ldots}}$$

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + 0.4142135...}}$$

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2.4142135\ldots}}}$$

$$\sqrt{\mathbf{2}} = \mathbf{1} + \cfrac{\mathbf{1}}{\mathbf{2} + \cfrac{\mathbf{1}}{\mathbf{2} + \cfrac{\mathbf{1}}{\mathbf{2} + \mathbf{0.4142135...}}}}$$

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2.4142135\ldots}}}}$$

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + 0.4142135...}}}}$$

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2.4142135\ldots}}}}}$$

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + 0.4142135\ldots}}}}}$$

$$\mathbf{4+\sqrt[7]{-1+\sqrt[23]{\frac{2\sqrt[3]{-37+\sqrt{6384+e^{\sin 47^\circ}}}+87}{76549-\cot 19^\circ}}}}$$

$$4+\sqrt[7]{-1+\sqrt[23]{\frac{2\sqrt[3]{-37+\sqrt{6384+e^{\sin 47^\circ}}}+87}{76549-\cot 19^\circ}}}$$

$$= 4.178364147...$$

$$4+\sqrt[7]{-1+\sqrt[23]{\frac{2\sqrt[3]{-37+\sqrt{6384+e^{\sin 47^\circ}}}+87}{76549-\cot 19^\circ}}}$$

$$= 4.178364147\ldots$$

$$= 4+\cfrac{1}{5+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{5+\cfrac{1}{1+\ldots}}}}}}}$$

$$\boldsymbol{\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \dots}}}}}}}}}$$

$$\boldsymbol{\pi \approx 3}$$

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \dots}}}}}}}}$$

$$\boldsymbol{\pi \approx 3 + \frac{1}{7}}$$

$$\boldsymbol{\pi \approx 3 + \frac{1}{7} = \frac{22}{7}}$$

$$\boldsymbol{\pi \approx 3 + \frac{1}{7} = \frac{22}{7} = 3.14...}$$

$$\boldsymbol{\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \dots}}}}}}}}}$$

$$\boldsymbol{\pi \approx 3 + \cfrac{1}{7 + \cfrac{1}{15}}}$$

$$\boldsymbol{\pi \approx 3 + \cfrac{1}{7 + \cfrac{1}{15}} = \frac{333}{106}}$$

$$\boldsymbol{\pi \approx 3 + \cfrac{1}{7 + \cfrac{1}{15}} = \frac{333}{106} = 3.1415...}$$

$$\boldsymbol{\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \dots}}}}}}}}}$$

$$\boldsymbol{\pi \approx 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1}}}}$$

$$\boldsymbol{\pi \approx 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1}}} = \frac{355}{113}}$$

$$\boldsymbol{\pi \approx 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1}}} = \frac{355}{113} = 3.141592...}$$

$$\boldsymbol{\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \dots}}}}}}}}}$$

$$\boldsymbol{\pi \approx 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292}}}}}$$

$$\boldsymbol{\pi \approx 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292}}}} = \frac{103993}{33102}}$$

$$\boldsymbol{\pi \approx 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292}}}} = \frac{103993}{33102} = 3.14159265\ldots}$$

$$\sqrt{3} = 1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \ldots}}}}}}}}$$

$$\sqrt{3} = 1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \dots}}}}}}}}$$

$$\frac{1}{1}, \frac{2}{1}, \frac{5}{3}, \frac{7}{4}, \frac{19}{11}, \frac{26}{15}, \frac{71}{41}, \frac{97}{56}, \frac{265}{153}, \frac{362}{209}, \frac{989}{571}, \dots$$

$$\boldsymbol{\xi} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cfrac{1}{a_5 + \cfrac{1}{a_6 + \cfrac{1}{a_7 + \cfrac{1}{a_8 + \ldots}}}}}}}}$$

$$\frac{p_1}{q_1},\ \frac{p_2}{q_2},\ \frac{p_3}{q_3},\ \frac{p_4}{q_4},\ \frac{p_5}{q_5},\ \frac{p_6}{q_6},\ \frac{p_7}{q_7},\ \frac{p_8}{q_8},\ \frac{p_9}{q_9},\ \frac{p_{10}}{q_{10}},\ldots$$

$$\xi = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cfrac{1}{a_5 + \cfrac{1}{a_6 + \cfrac{1}{a_7 + \cfrac{1}{a_8 + \dots}}}}}}}}$$

$$\frac{p_1}{q_1},\ \frac{p_2}{q_2},\ \frac{p_3}{q_3},\ \frac{p_4}{q_4},\ \frac{p_5}{q_5},\ \frac{p_6}{q_6},\ \frac{p_7}{q_7},\ \frac{p_8}{q_8},\ \frac{p_9}{q_9},\ \frac{p_{10}}{q_{10}}, \dots$$

$$\begin{aligned} p_i + a_i p_{i+1} &= p_{i+2} \\ q_i + a_i q_{i+1} &= q_{i+2} \end{aligned} \quad \text{and} \quad \begin{vmatrix} p_i & p_{i+1} \\ q_i & q_{i+1} \end{vmatrix} = (-1)^i$$

$$\xi = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cfrac{1}{a_5 + \cfrac{1}{a_6 + \cfrac{1}{a_7 + \cfrac{1}{a_8 + \ldots}}}}}}}}$$

$$\frac{p_1}{q_1},\ \frac{p_2}{q_2},\ \frac{p_3}{q_3},\ \frac{p_4}{q_4},\ \frac{p_5}{q_5},\ \frac{p_6}{q_6},\ \frac{p_7}{q_7},\ \frac{p_8}{q_8},\ \frac{p_9}{q_9},\ \frac{p_{10}}{q_{10}}, \ldots$$

$$\begin{aligned} p_i + a_i p_{i+1} &= p_{i+2} \\ q_i + a_i q_{i+1} &= q_{i+2} \end{aligned} \quad \text{and} \quad \begin{vmatrix} p_i & p_{i+1} \\ q_i & q_{i+1} \end{vmatrix} = (-1)^i$$

$$P_i(x) = q_i x - p_i,\ P_{i+1}(x) = q_{i+1} x - p_{i+1},\ P_{i+2}(x) = q_{i+2} x - p_{i+2}$$

$$\boldsymbol{\xi = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cfrac{1}{a_5 + \cfrac{1}{a_6 + \cfrac{1}{a_7 + \cfrac{1}{a_8 + \dots}}}}}}}}}$$

$$\boldsymbol{\frac{p_1}{q_1},\ \frac{p_2}{q_2},\ \frac{p_3}{q_3},\ \frac{p_4}{q_4},\ \frac{p_5}{q_5},\ \frac{p_6}{q_6},\ \frac{p_7}{q_7},\ \frac{p_8}{q_8},\ \frac{p_9}{q_9},\ \frac{p_{10}}{q_{10}}, \dots}$$

$$\boldsymbol{\begin{matrix} p_i + a_i p_{i+1} = p_{i+2} \\ q_i + a_i q_{i+1} = q_{i+2} \end{matrix} \quad \Longrightarrow \quad P_i(x) + a_i P_{i+1}(x) = P_{i+2}(x)}$$

$$\boldsymbol{P_i(x) = q_i x - p_i,\ P_{i+1}(x) = q_{i+1} x - p_{i+1},\ P_{i+2}(x) = q_{i+2} x - p_{i+2}}$$

THEOREM 10: **Let $a_1, a_2, a_3, \ldots \in Z^+$. Let**

$$P_1 = -1, \quad P_2 = x - a_0 \quad \text{and} \quad P_{i+2} = a_i P_{i+1}(x) + P_i(x)$$

for $i = 1, 2, \ldots$ Then

$$\lim_{n \to \infty} \alpha_n = \xi = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cfrac{1}{a_5 + \ldots}}}}}$$

where α_i are roots of P_i.

<u>THEOREM 10</u>: **Let $a_1, a_2, a_3, \ldots \in Z^+$. Let**

$$P_1 = -1, \quad P_2 = x - a_0 \quad \text{and} \quad P_{i+2} = a_i P_{i+1}(x) + P_i(x)$$

for $i = 1, 2, \ldots$ Then

$$\lim_{n\to\infty} \alpha_n = \xi = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cfrac{1}{a_5 + \ldots}}}}}$$

where α_i are roots of P_i. Moreover,

(i) $|P_1(\xi)| > |P_2(\xi)| > \ldots > |P_i(\xi)| > \ldots$

(ii) $\overline{|P_1|} < \overline{|P_2|} < \ldots < \overline{|P_i|} < \ldots$

(iii) **for any $P \in Z[x]$, $P \not\equiv 0$, with $|P(\xi)| < |P_i(\xi)|$ we have $\overline{|P|} \geq \overline{|P_{i+1}|}$.**

Example: **Let**

$$\begin{aligned}
P_1(x) &= -1 & & \\
P_2(x) &= x-1 & & \\
P_3(x) &= x-2 & \qquad P_3 &= P_2 + P_1 \\
P_4(x) &= 2x-3 & \qquad P_4 &= P_3 + P_2 \\
P_5(x) &= 3x-5 & \qquad P_5 &= P_4 + P_3 \\
P_6(x) &= 5x-8 & \qquad P_6 &= P_5 + P_4 \\
P_7(x) &= 8x-13 & \qquad P_7 &= P_6 + P_5 \\
P_8(x) &= 13x-21 & \qquad P_8 &= P_7 + P_6 \\
P_9(x) &= 21x-34 & \qquad P_9 &= P_8 + P_7
\end{aligned}$$

Then

$$\lim_{n\to\infty} \alpha_n = 1 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\ldots}}}}} = \frac{1+\sqrt{5}}{2}.$$

Example: **Let**

$$\begin{aligned}
P_1(x) &= -1 & & \\
P_2(x) &= x - 2 & & \\
P_3(x) &= x - 3 & P_3 &= 2P_2 + P_1 \\
P_4(x) &= 2x - 5 & P_4 &= 2P_3 + P_2 \\
P_5(x) &= 5x - 12 & P_5 &= 2P_4 + P_3 \\
P_6(x) &= 12x - 29 & P_6 &= 2P_5 + P_4 \\
P_7(x) &= 29x - 70 & P_7 &= 2P_6 + P_5 \\
P_8(x) &= 70x - 169 & P_8 &= 2P_7 + P_6 \\
P_9(x) &= 169x - 408 & P_9 &= 2P_8 + P_7
\end{aligned}$$

Then

$$\lim_{n\to\infty} \alpha_n = 2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \dots}}}}} = 1 + \sqrt{2}.$$

<u>Example</u>: Let

$$\begin{aligned}
P_1(x) &= -1 \\
P_2(x) &= x-2 \\
P_3(x) &= x-3 & P_3 &= P_2 + P_1 \\
P_4(x) &= 3x-8 & P_4 &= 2P_3 + P_2 \\
P_5(x) &= 4x-11 & P_5 &= P_4 + P_3 \\
P_6(x) &= 11x-30 & P_6 &= 2P_5 + P_4 \\
P_7(x) &= 15x-41 & P_7 &= P_6 + P_5 \\
P_8(x) &= 41x-112 & P_8 &= 2P_7 + P_6 \\
P_9(x) &= 56x-153 & P_9 &= P_8 + P_7
\end{aligned}$$

Then

$$\lim_{n\to\infty} \alpha_n = 2 + \cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\dots}}}}} = 1+\sqrt{3}.$$

EXAMPLE (Roy, 2001): Let

$$\begin{aligned}
P_1(x) &= -1 \\
P_2(x) &= x \\
P_3(x) &= 3x - 1 & P_3 &= 3P_2 + P_1 \\
P_4(x) &= 7x - 2 & P_4 &= 2P_3 + P_2 \\
P_5(x) &= 31x - 9 & P_5 &= 4P_4 + P_3 \\
P_6(x) &= 69x - 20 & P_6 &= 2P_5 + P_4 \\
P_7(x) &= 169x - 49 & P_7 &= 2P_6 + P_5 \\
P_8(x) &= 745x - 216 & P_8 &= 4P_7 + P_6 \\
P_9(x) &= 1659x - 481 & P_9 &= 2P_8 + P_7
\end{aligned}$$

Then

$$\lim_{n\to\infty} \alpha_n = \cfrac{1}{3 + \cfrac{1}{2 + \cfrac{1}{4 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{4 + \dots}}}}}}$$

<u>Theorem 10</u>: **Let $a_1, a_2, a_3, \ldots \in Z^+$. Let**

$$P_1 = -1, \quad P_2 = x - a_0 \quad \text{and} \quad P_{i+2} = a_i P_{i+1}(x) + P_i(x)$$

for $i = 1, 2, \ldots$ Then

$$\lim_{n\to\infty} \alpha_n = \xi = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cfrac{1}{a_5 + \ldots}}}}}$$

where α_i are roots of P_i. Moreover,

(i) $|P_1(\xi)| > |P_2(\xi)| > \ldots > |P_i(\xi)| > \ldots$

(ii) $\overline{|P_1|} < \overline{|P_2|} < \ldots < \overline{|P_i|} < \ldots$

(iii) **for any $P \in Z[x]$, $P \not\equiv 0$, with $|P(\xi)| < |P_i(\xi)|$ we have $\overline{|P|} \geq \overline{|P_{i+1}|}$.**

<u>THEOREM ???</u>: **Let $a_1, a_2, a_3, \ldots \in Z^+$. Let**

$$P_1 = ???, \quad P_2 = ???, \quad P_3 = ???$$

and

$$P_{i+3} = a_i P_{i+2}(x) + b_i P_{i+1}(x) + P_i(x)$$

for $i = 1, 2, \ldots$ Then

$$\lim_{n \to \infty} \alpha_n = \xi$$

where α_i are roots of P_i. Moreover,

(i) $|P_1(\xi)| > |P_2(\xi)| > \ldots > |P_i(\xi)| > \ldots$

(ii) $\overline{P_1} < \overline{P_2} < \ldots < \overline{P_i} < \ldots$

(iii) **for any $P \in Z[x]$, $\deg P \leq 2$, $P \not\equiv 0$,**

with $|P(\xi)| < |P_i(\xi)|$ we have $\overline{P} \geq \overline{P_{i+1}}$.

EXAMPLE: Let

$$
\begin{aligned}
&P_1(x) = 1 \\
&P_2(x) = x \\
&P_3(x) = x^2 \\
&P_4(x) = x^2 + x + 1 && P_4 = P_3 + P_2 + P_1 \\
&P_5(x) = 2x^2 + 2x + 1 && P_5 = P_4 + P_3 + P_2 \\
&P_6(x) = 4x^2 + 3x + 2 && P_6 = P_5 + P_4 + P_3 \\
&P_7(x) = 7x^2 + 6x + 4 && P_7 = P_6 + P_5 + P_4 \\
&P_8(x) = 13x^2 + 11x + 7 && P_8 = P_7 + P_6 + P_5 \\
&P_9(x) = 24x^2 + 20x + 13 && P_9 = P_8 + P_7 + P_6 \\
&P_{10}(x) = 44x^2 + 37x + 24 && P_{10} = P_9 + P_8 + P_7
\end{aligned}
$$

then

$$\lim_{n\to\infty} \alpha_n \approx -0.4196433776 - 0.6062907292I,$$

$$\lim_{n\to\infty} \beta_n \approx -0.4196433776 + 0.6062907292I$$

EXAMPLE: **Let**

$$\begin{aligned}
P_1(x) &= 1 \\
P_2(x) &= x \\
P_3(x) &= x^2 \\
P_4(x) &= x^2 - x - 1 & P_4 &= P_3 - P_2 - P_1 \\
P_5(x) &= -2x - 1 & P_5 &= P_4 - P_3 - P_2 \\
P_6(x) &= -2x^2 - x & P_6 &= P_5 - P_4 - P_3 \\
P_7(x) &= -3x^2 + 2x + 2 & P_7 &= P_6 - P_5 - P_4 \\
P_8(x) &= -x^2 + 5x + 3 & P_8 &= P_7 - P_6 - P_5 \\
P_9(x) &= 4x^2 + 4x + 1 & P_9 &= P_8 - P_7 - P_6 \\
P_{10}(x) &= 8x^2 - 3x - 4 & P_{10} &= P_9 - P_8 - P_7
\end{aligned}$$

then

$$\lim_{n\to\infty} \alpha_n \approx -0.5436,$$

$$\lim_{n\to\infty} \beta_n = ???$$

EXAMPLE: **Let**

$$
\begin{array}{ll}
P_1(x) = 1 & \\
P_2(x) = x & \\
P_3(x) = -x^2 & \\
P_4(x) = -x^2 - x - 1 & P_4 = P_3 - P_2 - P_1 \\
P_5(x) = -2x - 1 & P_5 = P_4 - P_3 - P_2 \\
P_6(x) = 2x^2 - x & P_6 = P_5 - P_4 - P_3 \\
P_7(x) = 3x^2 + 2x + 2 & P_7 = P_6 - P_5 - P_4 \\
P_8(x) = x^2 + 5x + 3 & P_8 = P_7 - P_6 - P_5 \\
P_9(x) = -4x^2 + 4x + 1 & P_9 = P_8 - P_7 - P_6 \\
P_{10}(x) = -8x^2 - 3x - 4 & P_{10} = P_9 - P_8 - P_7
\end{array}
$$

then

$$\lim_{n\to\infty} \alpha_n = ???,$$

$$\lim_{n\to\infty} \beta_n = ???$$

<u>EXAMPLE</u>: **Let**

$$
\begin{array}{ll}
P_1(x) = 1 & \\
P_2(x) = -x & \\
P_3(x) = -x^2 & \\
P_4(x) = -x^2 - x + 1 & P_4 = P_3 + P_2 + P_1 \\
P_5(x) = -2x^2 - 2x + 1 & P_5 = P_4 + P_3 + P_2 \\
P_6(x) = -4x^2 - 3x + 2 & P_6 = P_5 + P_4 + P_3 \\
P_7(x) = -7x^2 - 6x + 4 & P_7 = P_6 + P_5 + P_4 \\
P_8(x) = -13x^2 - 11x + 7 & P_8 = P_7 + P_6 + P_5 \\
P_9(x) = -24x^2 - 20x + 13 & P_9 = P_8 + P_7 + P_6 \\
P_{10}(x) = -44x^2 - 37x + 24 & P_{10} = P_9 + P_8 + P_7
\end{array}
$$

then

$$\xi_1 = \lim_{n\to\infty} \alpha_n = -1.2680... \quad \text{and} \quad \xi_2 = \lim_{n\to\infty} \beta_n = 0.4287...$$

where ξ_1 **and** ξ_2 **are roots of** $x^6 - 2x^5 + x^4 + 4x^3 - x^2 + 2x - 1$.

<u>EXAMPLE</u>: Let

$$\begin{aligned}
&P_1(x) = 1 && \\
&P_2(x) = -x && \\
&P_3(x) = -x^2 && \\
&P_4(x) = -x^2 - x + 1 && P_4 = P_3 + P_2 + P_1 \\
&P_5(x) = -2x^2 - 2x + 1 && P_5 = P_4 + P_3 + P_2 \\
&P_6(x) = -4x^2 - 3x + 2 && P_6 = P_5 + P_4 + P_3 \\
&P_7(x) = -7x^2 - 6x + 4 && P_7 = P_6 + P_5 + P_4 \\
&P_8(x) = -13x^2 - 11x + 7 && P_8 = P_7 + P_6 + P_5 \\
&P_9(x) = -24x^2 - 20x + 13 && P_9 = P_8 + P_7 + P_6 \\
&P_{10}(x) = -44x^2 - 37x + 24 && P_{10} = P_9 + P_8 + P_7
\end{aligned}$$

then

$$\xi_1 = \lim_{n\to\infty} \alpha_n = -1.2680... \quad \text{and} \quad \xi_2 = \lim_{n\to\infty} \beta_n = 0.4287...$$

where ξ_1 and ξ_2 are roots of $x^6-2x^5+x^4+4x^3-x^2+2x-1$. Moreover,

$$|P_i(\xi_1)| \sim \boxed{P_i}^{-0.5} \quad \text{and} \quad |P_i(\xi_2)| \sim \boxed{P_i}^{-0.5}.$$

EXAMPLE: **Let**

$$\begin{aligned}
P_1(x) &= -1 & & \\
P_2(x) &= x-2 & & \\
P_3(x) &= x-3 & P_3 &= P_2 + P_1 \\
P_4(x) &= 3x-8 & P_4 &= 2P_3 + P_2 \\
P_5(x) &= 4x-11 & P_5 &= P_4 + P_3 \\
P_6(x) &= 11x-30 & P_6 &= 2P_5 + P_4 \\
P_7(x) &= 15x-41 & P_7 &= P_6 + P_5 \\
P_8(x) &= 41x-112 & P_8 &= 2P_7 + P_6 \\
P_9(x) &= 56x-153 & P_9 &= P_8 + P_7
\end{aligned}$$

Then

$$\lim_{n\to\infty} \alpha_n = 2 + \cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\dots}}}}} = 1+\sqrt{3}.$$

<u>Example</u>: Let $\xi = 0.5436...$ be a root of $x^3 + x^2 + x - 1$. We have

$$
\begin{array}{ll}
P_1(x) = 1 & \\
P_2(x) = x & \\
P_3(x) = x^2 & \\
P_4(x) = -x^2 - x + 1 & P_4 = -P_3 - P_2 + P_1 \\
P_5(x) = -2x^2 + x & P_5 = -2P_3 + P_2 \\
P_6(x) = 3x^2 + 2x - 2 & P_6 = -2P_4 + P_3 \\
P_7(x) = -4x^2 + 4x - 1 & P_7 = P_6 + 3P_5 + P_4 \\
P_8(x) = -8x^2 - 3x + 4 & P_8 = -2P_6 + P_5 \\
P_9(x) = -5x^2 - 12x + 8 & P_9 = 2P_8 - 2P_7 + P_6 \\
P_{10}(x) = -7x^2 + 13x - 5 & P_{10} = -P_9 + P_8 + P_7 \\
P_{11}(x) = -20x^2 - 2x + 7 & P_{11} = P_9 + P_8 + P_7 \\
P_{12}(x) = -18x^2 - 27x + 20 & P_{12} = P_9 - P_8 + P_7 \\
P_{13}(x) = -9x^2 + 38x - 18 & P_{13} = -P_9 + P_8 + P_7 \\
P_{14}(x) = -47x^2 + 9x + 9 & P_{14} = P_9 + P_8 + P_7
\end{array}
$$

<u>Example</u>: Let $\xi = 0.5436...$ be a root of $x^3 + x^2 + x - 1$. We have

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\begin{array}{ll}
P_1(x) = 1 & \\
P_2(x) = x & \\
P_3(x) = x^2 & \\
P_4(x) = -x^2 - x + 1 & (-1, -1, 1) \\
P_5(x) = -2x^2 + x & (0, -2, 1) \\
P_6(x) = 3x^2 + 2x - 2 & (0, -2, 1) \\
P_7(x) = -4x^2 + 4x - 1 & (1, 3, 1) \\
P_8(x) = -8x^2 - 3x + 4 & (0, -2, 1) \\
P_9(x) = -5x^2 - 12x + 8 & (2, -2, 1) \\
P_{10}(x) = -7x^2 + 13x - 5 & (-1, 1, 1) \\
P_{11}(x) = -20x^2 - 2x + 7 & (1, 1, 1) \\
P_{12}(x) = -18x^2 - 27x + 20 & (1, -1, 1) \\
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P_7(x) = -4x^2 + 4x - 1 & (1,3,1) \\
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P_{13}(x) = -9x^2 + 38x - 18 & (1,1,1) \\
P_{14}(x) = -47x^2 + 9x + 9 & (1,1,1)
\end{array}
$$

$$(1,1,1)\ \ (1,1,0)\ \ (0,1,1)\ \ (1,2,1)\ \ (1,2,2)\ \ (1,2,0)\ \ (1,3,1)$$

EXAMPLE: Let $\xi = 1.6180...$ be a root of $x^2 - x - 1$. We have

$$\begin{array}{ll}
P_1(x) = -1 & \\
P_2(x) = x - 1 & \\
P_3(x) = x - 2 & (1,1) \\
P_4(x) = 2x - 3 & (1,1) \\
P_5(x) = 3x - 5 & (1,1) \\
P_6(x) = 5x - 8 & (1,1) \\
P_7(x) = 8x - 13 & (1,1) \\
P_8(x) = 13x - 21 & (1,1) \\
P_9(x) = 21x - 34 & (1,1) \\
P_{10}(x) = 34x - 55 & (1,1) \\
P_{11}(x) = 55x - 89 & (1,1) \\
P_{12}(x) = 89x - 144 & (1,1)
\end{array}$$

Example: Let $\xi = 0.5436...$ be a root of $x^3 + x^2 + x - 1$. We have

$$
\begin{aligned}
P_1(x) &= 1 & \\
P_2(x) &= x & \\
P_3(x) &= x^2 & \\
P_4(x) &= -x^2 - x + 1 & (1,1,1)\\
P_5(x) &= -2x^2 + x & (0,2,1)\\
P_6(x) &= 3x^2 + 2x - 2 & (0,2,1)\\
P_7(x) &= -4x^2 + 4x - 1 & (1,3,1)\\
P_8(x) &= -8x^2 - 3x + 4 & (0,2,1)\\
P_9(x) &= -5x^2 - 12x + 8 & (2,2,1)\\
P_{10}(x) &= -7x^2 + 13x - 5 & (1,1,1)\\
P_{11}(x) &= -20x^2 - 2x + 7 & (1,1,1)\\
P_{12}(x) &= -18x^2 - 27x + 20 & (1,1,1)\\
P_{13}(x) &= -9x^2 + 38x - 18 & (1,1,1)\\
P_{14}(x) &= -47x^2 + 9x + 9 & (1,1,1)
\end{aligned}
$$

$(1,1,1)$ $(1,1,0)$ $(0,1,1)$ $(1,2,1)$ $(1,2,2)$ $(1,2,0)$ $(1,3,1)$

<u>EXAMPLE</u>: **Let**

$$\begin{aligned}
&P_1(x) = 1 \\
&P_2(x) = -x \\
&P_3(x) = x^2 \\
&P_4(x) = x^2 + x - 1 && (1, -1, -1) \\
&P_5(x) = 2x - 1 && (1, -1, -1) \\
&P_6(x) = -2x^2 + x && (1, -1, -1) \\
&P_7(x) = -3x^2 - 2x + 2 && (1, -1, -1) \\
&P_8(x) = -x^2 - 5x + 3 && (1, -1, -1) \\
&P_9(x) = 4x^2 - 4x + 1 && (1, -1, -1) \\
&P_{10}(x) = 8x^2 + 3x - 4 && (1, -1, -1)
\end{aligned}$$

then

$$\xi = \lim_{n\to\infty} \alpha_n \approx 0.5436 \text{ (root of } x^3 + x^2 + x - 1),$$

$$\lim_{n\to\infty} \beta_n = ???$$

and $|P_i(\xi)| \sim \lceil P_i \rceil^{-2}$.

<u>Example</u>: **Let**

$$\begin{array}{ll}
P_1(x) = 1 & \\
P_2(x) = -x & \\
P_3(x) = x^2 & \\
P_4(x) = x^2 + x - 1 & P_4 = P_3 - P_2 - P_1 \\
P_5(x) = 2x - 1 & P_5 = P_4 - P_3 - P_2 \\
P_6(x) = -2x^2 + x & P_6 = P_5 - P_4 - P_3 \\
P_7(x) = -3x^2 - 2x + 2 & P_7 = P_6 - P_5 - P_4 \\
P_8(x) = -x^2 - 5x + 3 & P_8 = P_7 - P_6 - P_5 \\
P_9(x) = 4x^2 - 4x + 1 & P_9 = P_8 - P_7 - P_6 \\
P_{10}(x) = 8x^2 + 3x - 4 & P_{10} = P_9 - P_8 - P_7
\end{array}$$

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$$\xi = \lim_{n\to\infty} \alpha_n \approx 0.5436 \text{ (root of } x^3 + x^2 + x - 1),$$

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and $|P_i(\xi)| \sim \overline{P_i}^{-2}$.

Example: **Let**

$$\begin{array}{ll} P_1(x) = x & \\ P_2(x) = x^2 & \\ P_3(x) = -x+1 & \\ P_4(x) = -x^2+x & (1,-1,0) \\ P_5(x) = x^2+2x-2 & (1,-2,0) \\ P_6(x) = x^2-2x+1 & (1,-1,0) \\ P_7(x) = -3x^2-3x+4 & (1,-2,0) \\ P_8(x) = 4x-3 & (1,-1,0) \\ P_9(x) = 7x^2+4x-7 & (1,-2,0) \\ P_{10}(x) = -3x^2-7x+7 & (1,-1,0) \end{array}$$

then

$$\xi = \lim_{n\to\infty} \alpha_n \approx 0.7548 \text{ (root of } x^3+x^2-1),$$

$$\lim_{n\to\infty} \beta_n = ???$$

and $|P_i(\xi)| \sim \boxed{P_i}^{-2}$.

<u>Example</u> (Roy, 2001): Let

$$\begin{aligned}
P_1(x) &= -1 \\
P_2(x) &= x \\
P_3(x) &= 3x - 1 & P_3 &= 3P_2 + P_1 \\
P_4(x) &= 7x - 2 & P_4 &= 2P_3 + P_2 \\
P_5(x) &= 31x - 9 & P_5 &= 4P_4 + P_3 \\
P_6(x) &= 69x - 20 & P_6 &= 2P_5 + P_4 \\
P_7(x) &= 169x - 49 & P_7 &= 2P_6 + P_5 \\
P_8(x) &= 745x - 216 & P_8 &= 4P_7 + P_6 \\
P_9(x) &= 1659x - 481 & P_9 &= 2P_8 + P_7
\end{aligned}$$

Then

$$\lim_{n\to\infty} \alpha_n = \cfrac{1}{3 + \cfrac{1}{2 + \cfrac{1}{4 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{4 + \ldots}}}}}}$$

EXAMPLE (Roy, 2001): Let

$$
\begin{aligned}
P_1(x) &= -1 & & \\
P_2(x) &= x & & \\
P_3(x) &= 3x - 1 & & (3, 1) \\
P_4(x) &= 7x - 2 & & (2, 1) \\
P_5(x) &= 31x - 9 & & (4, 1) \\
P_6(x) &= 69x - 20 & & (2, 1) \\
P_7(x) &= 169x - 49 & & (2, 1) \\
P_8(x) &= 745x - 216 & & (4, 1) \\
P_9(x) &= 1659x - 481 & & (2, 1)
\end{aligned}
$$

Then

$$
\lim_{n\to\infty} \alpha_n = \cfrac{1}{3 + \cfrac{1}{2 + \cfrac{1}{4 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{4 + \ldots}}}}}}
$$

<u>Example</u>: **Let**

$$
\begin{aligned}
P_1(x) &= x \\
P_2(x) &= x^2 \\
P_3(x) &= -x + 1 \\
P_4(x) &= -x^2 + x && (1, -1, 0) \\
P_5(x) &= x^2 + 2x - 2 && (1, -2, 0) \\
P_6(x) &= x^2 - 2x + 1 && (1, -1, 0) \\
P_7(x) &= -3x^2 - 3x + 4 && (1, -2, 0) \\
P_8(x) &= 4x - 3 && (1, -1, 0) \\
P_9(x) &= 7x^2 + 4x - 7 && (1, -2, 0) \\
P_{10}(x) &= -3x^2 - 7x + 7 && (1, -1, 0)
\end{aligned}
$$

then

$$\xi = \lim_{n\to\infty} \alpha_n \approx 0.7548 \text{ (root of } x^3 + x^2 - 1\text{)},$$

$$\lim_{n\to\infty} \beta_n = ???$$

and $|P_i(\xi)| \sim \boxed{P_i}^{-2}$.