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REMARK 1: We call $\frac{1}{\sqrt{5}}$ the one-dimensional Diophantine approximation constant.

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$$\sup_{\xi \in \mathbb{R}} \liminf_{q \to \infty} |P(\xi)|\, p, \quad \sup_{\xi \in \mathbb{R}} \liminf_{q \to \infty} |P(\xi)| \sqrt{p^2 + q^2}, \quad \sup_{\xi \in \mathbb{R}} \liminf_{q \to \infty} |P(\xi)| \sqrt[n]{p^n + q^n}, \textit{etc.}$$

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Let ξ be a real number that is not rational nor quadratic irrational. What is

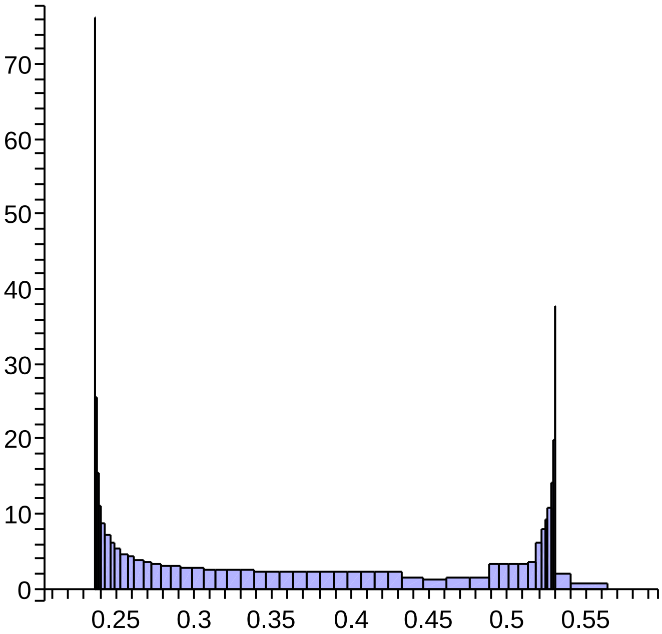
$$\sup_{\xi\in\mathbb{R}} \liminf_{P\in\mathbb{Z}[x]} |P(\xi)|\, \|P\|^2 \ ?$$

Here $P(x) = ax^2 + bx + c$ and $\|P\|$ is

$$\max(|a|,|b|,|c|) \quad \text{or} \quad \max(|a|,|b|) \quad \text{or} \quad \sqrt[k]{|a|^k+|b|^k+|c|^k} \quad \text{or} \quad |a| \ \textit{etc.}$$

Let $\xi = 0.5436...$ be a root of $f(x) = x^3 + x^2 + x - 1$. Then

1	$-x^2 - x + 1$	$[1, -1, -1]$	0.48214
2	$-2x^2 + x$	$[1, -2, 0]$	0.23753
3	$3x^2 + 2x - 2$	$[1, -2, 0]$	0.43909
4	$-4x^2 + 4x - 1$	$[1, 3, 1]$	0.25195
5	$-8x^2 - 3x + 4$	$[1, -2, 0]$	0.36944
6	$-5x^2 - 12x + 8$	$[1, -2, 2]$	0.52585
7	$-7x^2 + 13x - 5$	$[1, 1, -1]$	0.29817
8	$-20x^2 - 2x + 7$	$[1, 1, 1]$	0.30221
9	$-18x^2 - 27x + 20$	$[1, -1, 1]$	0.52701
10	$-9x^2 + 38x - 18$	$[1, 1, -1]$	0.36462
11	$-47x^2 + 9x + 9$	$[1, 1, 1]$	0.25421
12	$-56x^2 - 56x + 47$	$[1, -1, 1]$	0.49437
13	$-103x^2 + 56x$	$[1, 2, 0]$	0.23684
14	$-159x^2 - 103x + 103$	$[1, 2, 0]$	0.43562
15	$-206x^2 + 215x - 56$	$[1, 3, -1]$	0.25421
16	$-421x^2 - 150x + 206$	$[1, 2, 0]$	0.36462
17	$271x^2 + 627x - 421$	$[1, 2, -2]$	0.52701
18	$-356x^2 + 692x - 271$	$[1, 1, 1]$	0.30221
19	$-1048x^2 - 85x + 356$	$[1, -1, 1]$	0.29817
20	$963x^2 + 1404x - 1048$	$[1, 1, -1]$	0.52582



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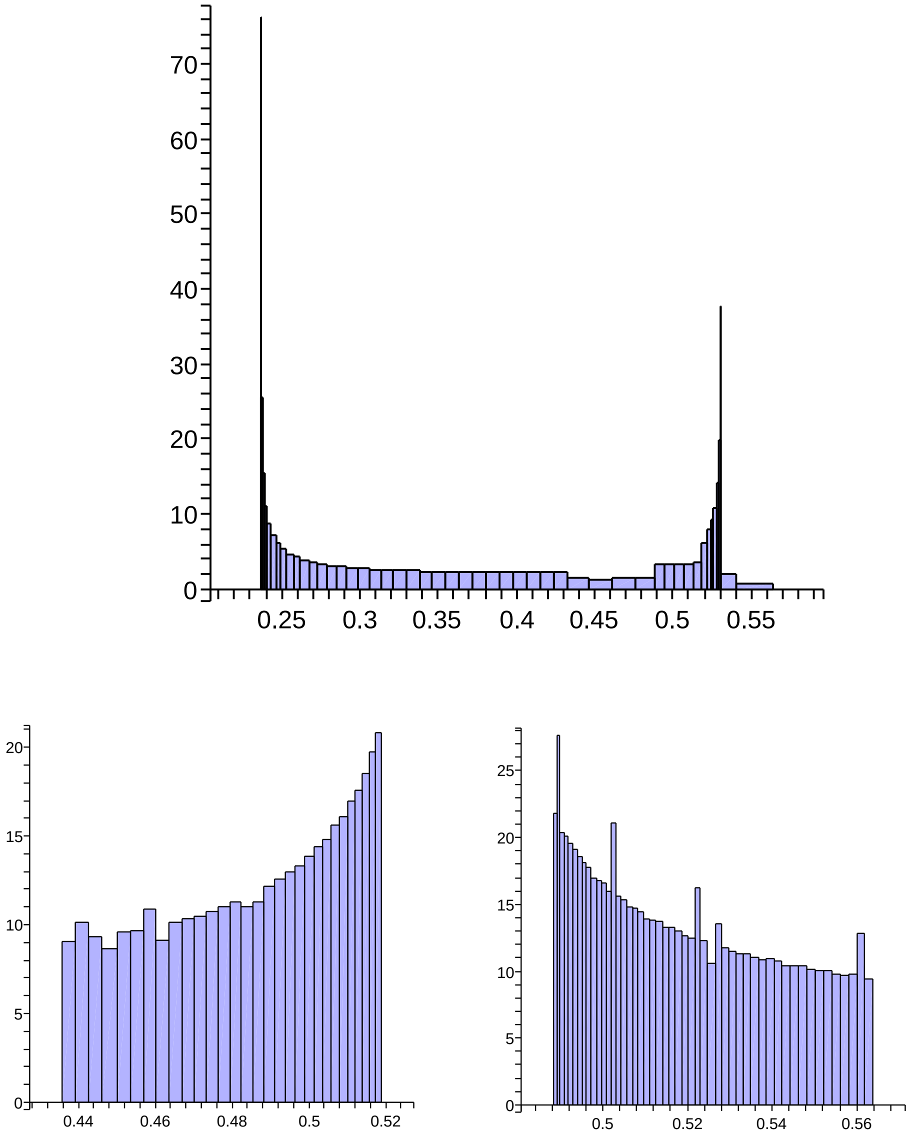
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5	$-8\,x^2 - 3\,x + 4$	$[1, -2, 0]$	$-x$	0.36944
6	$-5\,x^2 - 12\,x + 8$	$[1, -2, 2]$	$-x$	0.52585
7	$-7\,x^2 + 13\,x - 5$	$[1, 1, -1]$	x	0.29817
8	$-20\,x^2 - 2\,x + 7$	$[1, 1, 1]$	$-x$	0.30221
9	$-18\,x^2 - 27\,x + 20$	$[1, -1, 1]$	$-x$	0.52701
10	$-9\,x^2 + 38\,x - 18$	$[1, 1, -1]$	x	0.36462
11	$-47\,x^2 + 9\,x + 9$	$[1, 1, 1]$	$-x$	0.25421
12	$-56\,x^2 - 56\,x + 47$	$[1, -1, 1]$	$-x$	0.49437
13	$-103\,x^2 + 56\,x$	$[1, 2, 0]$	$-x^2$	0.23684
14	$-159\,x^2 - 103\,x + 103$	$[1, 2, 0]$	$-x$	0.43562
15	$-206\,x^2 + 215\,x - 56$	$[1, 3, -1]$	$-x^2$	0.25421
16	$-421\,x^2 - 150\,x + 206$	$[1, 2, 0]$	$-x$	0.36462
17	$271\,x^2 + 627\,x - 421$	$[1, 2, -2]$	x	0.52701
18	$-356\,x^2 + 692\,x - 271$	$[1, 1, 1]$	$-x$	0.30221
19	$-1048\,x^2 - 85\,x + 356$	$[1, -1, 1]$	$-x$	0.29817
20	$963\,x^2 + 1404\,x - 1048$	$[1, 1, -1]$	x	0.52582

21	$-441\,x^2+2011\,x-963$	$[1,1,1]$	$-x$	0.36950
22	$-2452\,x^2+522\,x+441$	$[1,-1,1]$	$-x$	0.25197
23	$-2893\,x^2+2533\,x-522$	$[0,1,1]$	$-x^2-x$	0.49147
24	$2974\,x^2+2893\,x-2452$	$[1,1,-1]$	$-1/2\,x^2-1/2$	0.49108
25	$-5345\,x^2+3055\,x-81$	$[1,1,0]$	x^2	0.23692
26	$8400\,x^2+5264\,x-5345$	$[1,2,-1]$	x	0.43099
27	$10609\,x^2-11536\,x+3136$	$[1,-3,-1]$	$-x^2$	0.25660
28	$-22145\,x^2-7473\,x+10609$	$[1,-2,0]$	x	0.35977
29	$-14672\,x^2-32754\,x+22145$	$[1,2,2]$	$-x$	0.52805
30	$18082\,x^2-36817\,x+14672$	$[1,-1,1]$	$-x$	0.30634
31	$-54899\,x^2-3410\,x+18082$	$[1,1,-1]$	x	0.29423
32	$-51489\,x^2-72981\,x+54899$	$[1,1,1]$	$-x$	0.52447
33	$21492\,x^2-106388\,x+51489$	$[1,-1,1]$	$-x$	0.37440
34	$-127880\,x^2+29997\,x+21492$	$[1,1,-1]$	x	0.24987
35	$-149372\,x^2+136385\,x-29997$	$[0,-1,1]$	$-x^2-x$	0.49498
36	$-157877\,x^2-149372\,x+127880$	$[1,1,1]$	$1/2\,x^2+1/2$	0.48766
37	$-277252\,x^2+166382\,x-8505$	$[1,1,0]$	$-x^2$	0.23717
38	$443634\,x^2+268747\,x-277252$	$[1,-2,-1]$	x	0.42632
39	$-545999\,x^2+618521\,x-174887$	$[1,3,1]$	x^2	0.25913
40	$-1164520\,x^2-371112\,x+545999$	$[1,-2,0]$	$-x$	0.35494

Let $\xi = 0.6180...$ be a root of $f(x) = x^2 + x - 1$. Then

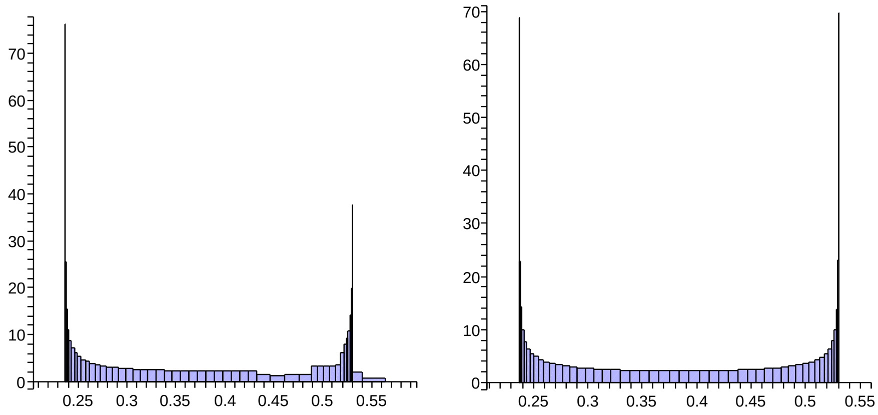
-1	-1			-1
0	x	$-x$		0.61803
1	$x - 1$	$-x$	[1, 1, 1]	0.54018
2	$2\,x - 1$	$-x$	[1, 1, 1]	0.47214
3	$3\,x - 2$	$-x$	[1, 1, 1]	0.43769
4	$5\,x - 3$	$-x$	[1, 1, 1]	0.45085
5	$8\,x - 5$	$-x$	[1, 1, 1]	0.44582
6	$13\,x - 8$	$-x$	[1, 1, 1]	0.44774
7	$21\,x - 13$	$-x$	[1, 1, 1]	0.44701
8	$34\,x - 21$	$-x$	[1, 1, 1]	0.44729
9	$55\,x - 34$	$-x$	[1, 1, 1]	0.44718
10	$89\,x - 55$	$-x$	[1, 1, 1]	0.44722
11	$144\,x - 89$	$-x$	[1, 1, 1]	0.44721
12	$233\,x - 144$	$-x$	[1, 1, 1]	0.44722
13	$377\,x - 233$	$-x$	[1, 1, 1]	0.44721
14	$610\,x - 377$	$-x$	[1, 1, 1]	0.44721
15	$987\,x - 610$	$-x$	[1, 1, 1]	0.44721
16	$1597\,x - 987$	$-x$	[1, 1, 1]	0.44721
17	$2584\,x - 1597$	$-x$	[1, 1, 1]	0.44721
18	$4181\,x - 2584$	$-x$	[1, 1, 1]	0.44721
19	$6765\,x - 4181$	$-x$	[1, 1, 1]	0.44721
20	$10946\,x - 6765$	$-x$	[1, 1, 1]	0.44721

Let $\xi = 0.5436...$ be a root of $f(x) = x^3 + x^2 + x - 1$. Then

1	$-x^2 - x + 1$	$[1, -1, -1]$	x	0.48214
2	$-2x^2 + x$	$[1, -2, 0]$	$-x^2$	0.23753
3	$3x^2 + 2x - 2$	$[1, -2, 0]$	x	0.43909
4	$-4x^2 + 4x - 1$	$[1, 3, 1]$	x^2	0.25195
5	$-8x^2 - 3x + 4$	$[1, -2, 0]$	$-x$	0.36944
6	$-5x^2 - 12x + 8$	$[1, -2, 2]$	$-x$	0.52585
7	$-7x^2 + 13x - 5$	$[1, 1, -1]$	x	0.29817
8	$-20x^2 - 2x + 7$	$[1, 1, 1]$	$-x$	0.30221
9	$-18x^2 - 27x + 20$	$[1, -1, 1]$	$-x$	0.52701
10	$-9x^2 + 38x - 18$	$[1, 1, -1]$	x	0.36462
11	$-47x^2 + 9x + 9$	$[1, 1, 1]$	$-x$	0.25421
12	$-56x^2 - 56x + 47$	$[1, -1, 1]$	$-x$	0.49437
13	$-103x^2 + 56x$	$[1, 2, 0]$	$-x^2$	0.23684
14	$-159x^2 - 103x + 103$	$[1, 2, 0]$	$-x$	0.43562
15	$-206x^2 + 215x - 56$	$[1, 3, -1]$	$-x^2$	0.25421
16	$-421x^2 - 150x + 206$	$[1, 2, 0]$	$-x$	0.36462
17	$271x^2 + 627x - 421$	$[1, 2, -2]$	x	0.52701
18	$-356x^2 + 692x - 271$	$[1, 1, 1]$	$-x$	0.30221
19	$-1048x^2 - 85x + 356$	$[1, -1, 1]$	$-x$	0.29817
20	$963x^2 + 1404x - 1048$	$[1, 1, -1]$	x	0.52582

Let $\xi = 0.5436...$ be a root of $f(x) = x^3 + x^2 + x - 1$. Then

1	x	$[1, -1, -1, 1]$	x	0.54369
2	x^2	$[1, -1, -1, 1]$	x	0.29560
3	$-x^2 - x + 1$	$[1, -1, -1, 1]$	x	0.48214
4	$2x - 1$	$[1, -1, -1, 1]$	x	0.43689
5	$2x^2 - x$	$[1, -1, -1, 1]$	x	0.23753
6	$-3x^2 - 2x + 2$	$[1, -1, -1, 1]$	x	0.43909
7	$x^2 + 5x - 3$	$[1, -1, -1, 1]$	x	0.49150
8	$4x^2 - 4x + 1$	$[1, -1, -1, 1]$	x	0.25195
9	$-8x^2 - 3x + 4$	$[1, -1, -1, 1]$	x	0.36944
10	$5x^2 + 12x - 8$	$[1, -1, -1, 1]$	x	0.52585
11	$7x^2 - 13x + 5$	$[1, -1, -1, 1]$	x	0.29817
12	$-20x^2 - 2x + 7$	$[1, -1, -1, 1]$	x	0.30221
13	$18x^2 + 27x - 20$	$[1, -1, -1, 1]$	x	0.52701
14	$9x^2 - 38x + 18$	$[1, -1, -1, 1]$	x	0.36462
15	$-47x^2 + 9x + 9$	$[1, -1, -1, 1]$	x	0.25421
16	$56x^2 + 56x - 47$	$[1, -1, -1, 1]$	x	0.49437
17	$-103x + 56$	$[1, -1, -1, 1]$	x	0.43562
18	$-103x^2 + 56x$	$[1, -1, -1, 1]$	x	0.23684
19	$159x^2 + 103x - 103$	$[1, -1, -1, 1]$	x	0.43562
20	$-56x^2 - 262x + 159$	$[1, -1, -1, 1]$	x	0.49437



0
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0.45
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0.55
0
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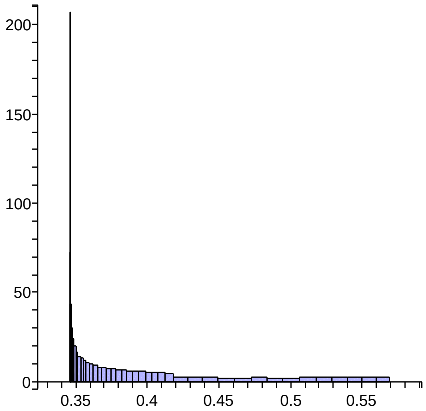
60
70
0.25
0.3
0.35
0.4
0.45
0.5
0.55

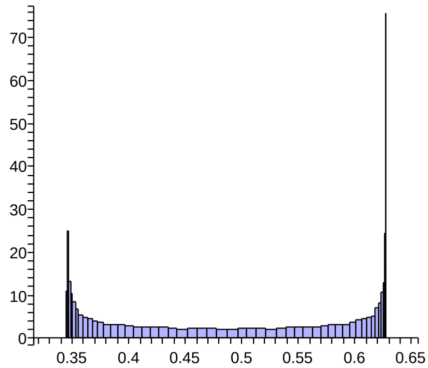
Let $\xi = 0.5436...$ be a root of $f(x) = x^3 + x^2 + x - 1$. Then

1	$-x^2 - x + 1$	$-x^2 - x + 1$
2	$-2x^2 + x$	$2x - 1$
3	$3x^2 + 2x - 2$	$2x^2 - x$
4	$-4x^2 + 4x - 1$	$-3x^2 - 2x + 2$
5	$-8x^2 - 3x + 4$	$x^2 + 5x - 3$
6	$-5x^2 - 12x + 8$	$4x^2 - 4x + 1$
7	$-7x^2 + 13x - 5$	$-8x^2 - 3x + 4$
8	$-20x^2 - 2x + 7$	$5x^2 + 12x - 8$
9	$-18x^2 - 27x + 20$	$7x^2 - 13x + 5$
10	$-9x^2 + 38x - 18$	$-20x^2 - 2x + 7$
11	$-47x^2 + 9x + 9$	$18x^2 + 27x - 20$
12	$-56x^2 - 56x + 47$	$9x^2 - 38x + 18$
13	$-103x^2 + 56x$	$-47x^2 + 9x + 9$
14	$-159x^2 - 103x + 103$	$56x^2 + 56x - 47$
15	$-206x^2 + 215x - 56$	$-103x + 56$
16	$-421x^2 - 150x + 206$	$-103x^2 + 56x$
17	$271x^2 + 627x - 421$	$159x^2 + 103x - 103$
18	$-356x^2 + 692x - 271$	$-56x^2 - 262x + 159$
19	$-1048x^2 - 85x + 356$	$-206x^2 + 215x - 56$
20	$963x^2 + 1404x - 1048$	$421x^2 + 150x - 206$

Let $\xi = 0.6823\ldots$ be a root of $x^3 + x - 1$. Then

1	$x^2 + x - 1$	$x - 1$
2	$2x^2 - 1$	$x^2 - x$
3	$2x^2 + 3x - 3$	$x^2 + x - 1$
4	$5x^2 + x - 3$	$x^2 - 2x + 1$
5	$8x^2 - 4x - 1$	$2x^2 - 1$
6	$4x^2 + 9x - 8$	$3x - 2$
7	$12x^2 + 5x - 9$	$3x^2 - 2x$
8	$21x^2 - 7x - 5$	$2x^2 + 3x - 3$
9	$7x^2 + 26x - 21$	$3x^2 - 5x + 2$
10	$28x^2 + 19x - 26$	$5x^2 + x - 3$
11	$54x^2 - 9x - 19$	$x^2 - 8x + 5$
12	$63x^2 + 64x - 73$	$8x^2 - 4x - 1$
13	$136x^2 + x - 64$	$4x^2 + 9x - 8$
14	$200x^2 - 135x - 1$	$9x^2 - 12x + 4$
15	$135x^2 + 201x - 200$	$12x^2 + 5x - 9$
16	$335x^2 + 66x - 201$	$5x^2 - 21x + 12$
17	$536x^2 - 269x - 66$	$21x^2 - 7x - 5$
18	$269x^2 + 602x - 536$	$7x^2 + 26x - 21$
19	$805x^2 + 333x - 602$	$26x^2 - 28x + 7$
20	$1407x^2 - 472x - 333$	$28x^2 + 19x - 26$
21	$472x^2 + 1740x - 1407$	$19x^2 - 54x + 28$
22	$1879x^2 + 1268x - 1740$	$54x^2 - 9x - 19$
23	$3619x^2 - 611x - 1268$	$9x^2 + 73x - 54$
24	$4230x^2 + 4276x - 4887$	$73x^2 - 63x + 9$
25	$9117x^2 + 46x - 4276$	$63x^2 + 64x - 73$


200
150
100
50
0
0.35
0.4
0.45
0.5
0.55


70
60
50
40
30
20
10
0
0.35
0.4
0.45
0.5
0.55
0.6
0.65

Consider a polynomial $f(x) = x^3 + a_2x^2 + a_1x - 1$ in $\mathbb{Z}[x]$ irreducible over $\mathbb{Q}$. Let ξ, ξ_1 and ξ_2 be its roots. Assume that ξ is real and ξ_1, ξ_2 are complex. We also assume that $|\xi| < 1$.

We introduce a sequence of polynomials $P_n(x) = a_n + b_nx + c_nx^2$ in $\mathbb{Z}[x]$, defined by $P_n(x) = x^n \mod f$. Put

$$K_n = |P_n(\xi)|(a_n^2 + b_n^2 + c_n^2)$$

and $I = \inf_n K_n, \quad S = \sup_n K_n, \quad \tilde{K_n} = (K_n - I)/S.$

<u>Theorem (Hensley-T.)</u>: A frequency curve for $\{\tilde{K_n}\}$ is $f(x) = \dfrac{1}{\pi\sqrt{x(1-x)}}$. Moreover, $I = \inf_n K_n$ and $S = \sup_n K_n$ are roots of $p(x) = c_6x^6 + \ldots c_2x^2 + c_1x + c_0$ with

$$c_6 = d^3$$

$$c_5 = 4\left({a_2}^4 - {a_2}^3 - 4\,{a_2}^2a_1 + {a_2}^2 + 4\,a_2a_1 + {a_1}^2 - 6\,a_2 - 3\,a_1 + 9\right)d^2$$

$$c_4 = -4\left({a_2}^4 - 4\,{a_2}^2a_1 + {a_2}^2 + 2\,{a_1}^2 - 8\,a_2 - 2\,a_1 + 3\right)d^2$$

$$c_3 = -16\left({a_2}^3{a_1}^2 + 6\,{a_2}^4 + 6\,{a_2}^2{a_1}^2 - 4\,a_2{a_1}^3 + 2\,{a_1}^4 + 4\,{a_2}^3 - 18\,{a_2}^2a_1 - 4\,{a_1}^3 + 10\,{a_2}^2 \right.$$
$$\left. -10\,a_2a_1 + 10\,{a_1}^2 - 27\,a_2\right)d$$

$$c_2 = 16\left(2\,{a_2}^4 + 5\,{a_2}^2{a_1}^2 + {a_1}^4 + 2\,{a_2}^3 - 2\,{a_1}^3 + 8\,{a_2}^2 + {a_1}^2 - 6\,a_2\right)d$$

$$c_1 = 64\,{a_2}^2{a_1}^4 + 64\,{a_1}^6 + 320\,{a_2}^3{a_1}^2 + 64\,{a_2}^2{a_1}^3 + 384\,a_2{a_1}^4 - 64\,{a_1}^5 + 320\,{a_2}^4 + 256\,{a_2}^3a_1$$
$$+704\,{a_2}^2{a_1}^2 - 384\,a_2{a_1}^3 + 128\,{a_1}^4 + 576\,{a_2}^3 - 384\,{a_2}^2a_1 + 576\,a_2{a_1}^2 + 192\,{a_1}^3 + 1152\,{a_2}^2$$

$$c_0 = -64\left({a_2}^2 + a_2a_1 + {a_1}^2 + 2\,a_2 - 2\,a_1 + 4\right)\left({a_2}^2 - a_2a_1 + {a_1}^2\right),$$

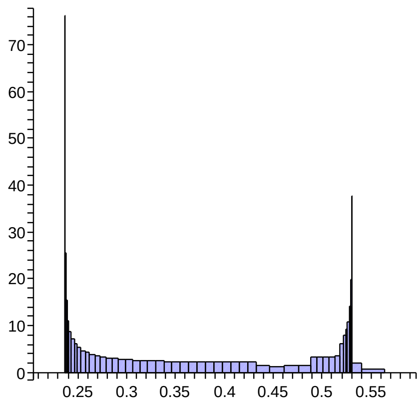
where $d = -4a_1^3 + 4a_2^3 + a_1^2a_2^2 - 18a_1a_2 - 27$ is the discriminant of f.

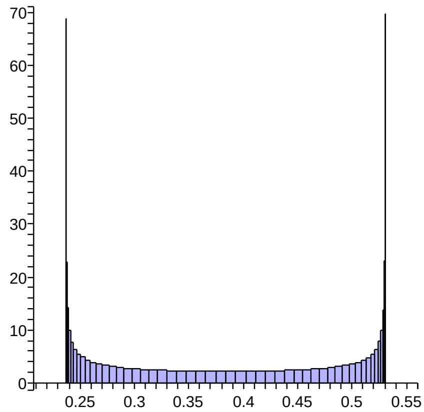
If $f(x) = x^3 + x^2 + x - 1$, then

$$I = \inf_n K_n = 0.2368.. \quad \text{and} \quad S = \sup_n K_n = 0.5308..$$

are roots of

$$p(x) = \left(11\,x^3 + 11\,x^2 + x - 1\right)\left(121\,x^3 - 143\,x^2 + 55\,x - 7\right)$$



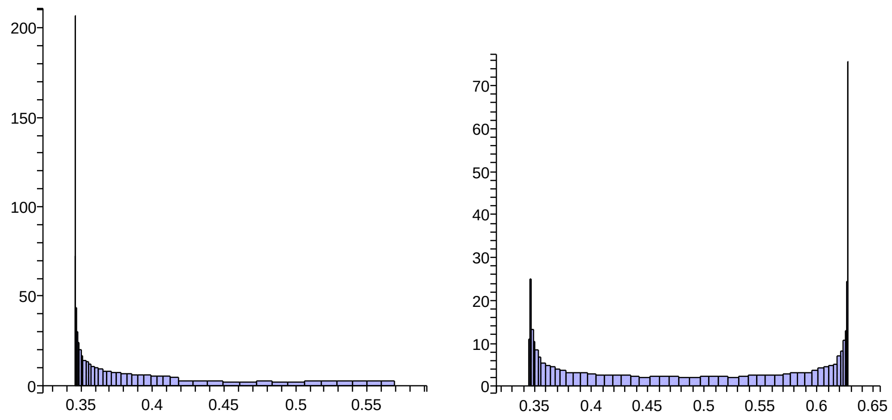


If $f(x) = x^3 + x - 1$, then

$$I = \inf_n K_n = 0.3458.. \quad \text{and} \quad S = \sup_n K_n = 0.6276..$$

are roots of

$$p(x) = 29791\,x^6 - 26908\,x^5 + 11532\,x^4 - 3968\,x^3 - 320\,x + 192$$



Consider a polynomial $f(x) = x^3 + a_2x^2 + a_1x - 1$ in $\mathbb{Z}[x]$ irreducible over $\mathbb{Q}$. Let ξ, ξ_1 and ξ_2 be its roots. Assume that ξ is real and ξ_1, ξ_2 are complex. We also assume that $|\xi| < 1$.

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$$K_n = |P_n(\xi)|(a_n^2 + b_n^2 + c_n^2)$$

and $I = \inf_n K_n, \quad S = \sup_n K_n, \quad \tilde{K}_n = (K_n - I)/S.$

THEOREM (HENSLEY-T.): A frequency curve for $\{\tilde{K}_n\}$ is $f(x) = \dfrac{1}{\pi\sqrt{x(1-x)}}$. Moreover, $I = \inf_n K_n$ and $S = \sup_n K_n$ are roots of $p(x) = c_6x^6 + \dots c_2x^2 + c_1x + c_0$ with

$$c_6 = d^3$$
$$c_5 = 4\left(a_2^4 - a_2^3 - 4a_2^2a_1 + a_2^2 + 4a_2a_1 + a_1^2 - 6a_2 - 3a_1 + 9\right)d^2$$
$$c_4 = -4\left(a_2^4 - 4a_2^2a_1 + a_2^2 + 2a_1^2 - 8a_2 - 2a_1 + 3\right)d^2$$
$$c_3 = -16\left(a_2^3a_1^2 + 6a_2^4 + 6a_2^2a_1^2 - 4a_2a_1^3 + 2a_1^4 + 4a_2^3 - 18a_2^2a_1 - 4a_1^3 + 10a_2^2 - 10a_2a_1 + 10a_1^2 - 27a_2\right)d$$
$$c_2 = 16\left(2a_2^4 + 5a_2^2a_1^2 + a_1^4 + 2a_2^3 - 2a_1^3 + 8a_2^2 + a_1^2 - 6a_2\right)d$$
$$c_1 = 64a_2^2a_1^4 + 64a_1^6 + 320a_2^3a_1^2 + 64a_2^2a_1^3 + 384a_2a_1^4 - 64a_1^5 + 320a_2^4 + 256a_2^3a_1 + 704a_2^2a_1^2 - 384a_2a_1^3 + 128a_1^4 + 576a_2^3 - 384a_2^2a_1 + 576a_2a_1^2 + 192a_1^3 + 1152a_2^2$$
$$c_0 = -64\left(a_2^2 + a_2a_1 + a_1^2 + 2a_2 - 2a_1 + 4\right)\left(a_2^2 - a_2a_1 + a_1^2\right),$$

where $d = -4a_1^3 + 4a_2^3 + a_1^2a_2^2 - 18a_1a_2 - 27$ is the discriminant of f.

Consider a polynomial $f(x) = x^3 + a_2x^2 + a_1x - 1$ in $\mathbb{Z}[x]$ irreducible over $\mathbb{Q}$. Let ξ, ξ_1 and ξ_2 be its roots. Assume that ξ is real and ξ_1, ξ_2 are complex. We also assume that $|\xi| < 1$.

We introduce a sequence of polynomials $P_n(x) = a_n + b_nx + c_nx^2$ in $\mathbb{Z}[x]$, defined by $P_n(x) = x^n \mod f$. Since $f(\xi) = f(\xi_1) = f(\xi_2) = 0$, we have

$$\begin{aligned} P_n(\xi) &= \xi^n = a_n + b_n\xi + c_n\xi^2, \\ P_n(\xi_1) &= \xi_1^n = a_n + b_n\xi_1 + c_n\xi_1^2, \\ P_n(\xi_2) &= \xi_2^n = a_n + b_n\xi_2 + c_n\xi_2^2. \end{aligned} \tag{1}$$

In other words,

$$\begin{pmatrix} \xi^n \\ \xi_1^n \\ \xi_2^n \end{pmatrix} = V \begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix}, \quad \text{where} \quad V = \begin{pmatrix} 1 & \xi & \xi^2 \\ 1 & \xi_1 & \xi_1^2 \\ 1 & \xi_2 & \xi_2^2 \end{pmatrix} \tag{2}$$

is the Vandermonde matrix. From (2) it follows that

$$\begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} = V^{-1} \begin{pmatrix} \xi^n \\ \xi_1^n \\ \xi_2^n \end{pmatrix}. \tag{3}$$

Since ξ, ξ_1 and ξ_2 are roots of $f(x)$ and the last coefficient of f is -1, it follows that $|\xi\xi_1\xi_2| = 1$. Therefore

$$|\xi|^{-1/2} = |\xi_1| = |\xi_2|, \tag{4}$$

because ξ_1 and ξ_2 are complex conjugates. Applying (4) to (3), we get

$$\begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} = V^{-1}\xi^{-n/2} \begin{pmatrix} \xi^{3n/2} \\ \cos(n\beta) + i\sin(n\beta) \\ \cos(n\beta) - i\sin(n\beta) \end{pmatrix}, \tag{5}$$

where $\beta = \arg(\xi_1)$. Observe that for large n, $\xi^{3n/2}$ is much smaller than $|\cos(n\beta) \pm i\sin(n\beta)|$, therefore

$$\begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} \approx V^{-1}\xi^{-n/2} \begin{pmatrix} 0 \\ \cos(n\beta) + i\sin(n\beta) \\ \cos(n\beta) - i\sin(n\beta) \end{pmatrix},$$

which can be rewritten as

$$\xi^{n/2} \begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} \approx \frac{2i}{D} \begin{pmatrix} \xi^{3/2}\sin((n-1)\beta) - \sin((n-2)\beta) \\ \xi^{-1}\sin((n-2)\beta) - \xi^2\sin(n\beta) \\ \xi\sin(n\beta) - \xi^{-1/2}\sin((n-1)\beta) \end{pmatrix}, \tag{6}$$

where D is the determinant of V.

Put

$$g_1(n) = \xi^{3/2}\sin((n-1)\beta) - \sin((n-2)\beta),$$

$$g_2(n) = \xi^{-1}\sin((n-2)\beta) - \xi^2\sin(n\beta),$$

$$g_3(n) = \xi\sin(n\beta) - \xi^{-1/2}\sin((n-1)\beta).$$

From (1) and (6) it follows that

$$\begin{aligned}(P_n(\xi))^{1/2}a_n &\approx \frac{2i}{D}g_1(n),\\ (P_n(\xi))^{1/2}b_n &\approx \frac{2i}{D}g_2(n),\\ (P_n(\xi))^{1/2}c_n &\approx \frac{2i}{D}g_3(n).\end{aligned} \tag{7}$$

We have

$$(P_n(\xi))^{1/2}a_n \approx A_1\sin(n\beta) + B_1\cos(n\beta)$$

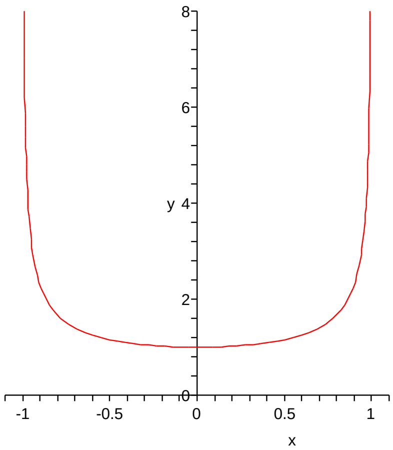
$$(P_n(\xi))^{1/2}b_n \approx A_2\sin(n\beta) + B_2\cos(n\beta)$$

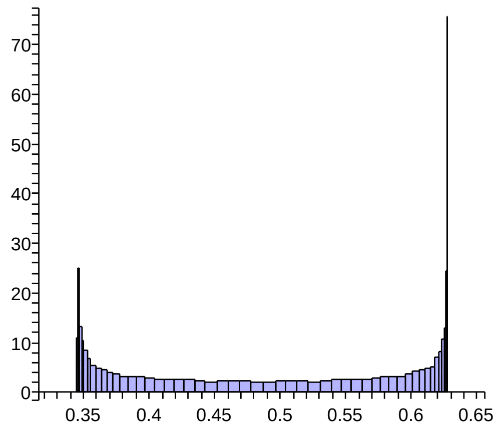
$$(P_n(\xi))^{1/2}c_n \approx A_3\sin(n\beta) + B_3\cos(n\beta)$$

The density function is

$$\varphi(x) = \frac{1}{\sqrt{A_i^2 + B_i^2 - x^2}}, \quad i = 1, 2, 3.$$

If $A_i^2 + B_i^2 = 1$,





Put

$$g_1(n) = \xi^{3/2}\sin((n-1)\beta) - \sin((n-2)\beta),$$

$$g_2(n) = \xi^{-1}\sin((n-2)\beta) - \xi^2\sin(n\beta),$$

$$g_3(n) = \xi\sin(n\beta) - \xi^{-1/2}\sin((n-1)\beta).$$

From (1) and (6) it follows that

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To show that the points $(g_1(n), g_2(n), g_3(n))$ trace an ellipse in 3-dimensional space, we note that

$$g_1(n) + \xi g_2(n) + \xi^2 g_3(n) = 0$$

and

$$-\frac{g_1^2(n)}{a^2} + \frac{g_2^2(n)}{b^2} + \frac{g_3^2(n)}{c^2} = 1,$$

where

$$a^2 = \frac{\xi^{3/2}\sin^2\beta(\xi^3 + 1 - 2\xi^{3/2}\cos\beta)}{\xi^{3/2} + \cos\beta},$$

$$b^2 = -\frac{\sin^2\beta(\xi^3 + 1 - 2\xi^{3/2}\cos\beta)}{\xi^{1/2}\cos\beta},$$

$$c^2 = \frac{\sin^2\beta(\xi^3 + 1 - 2\xi^{3/2}\cos\beta)}{\xi(1 + \xi^{3/2}\cos\beta)}.$$

We now find

$$\inf_n |P_n(\xi)|(a_n^2 + b_n^2 + c_n^2) \quad \text{and} \quad \sup_n |P_n(\xi)|(a_n^2 + b_n^2 + c_n^2).$$

Put

$$I = \inf_n |P_n(\xi)|(a_n^2 + b_n^2 + c_n^2) \quad \text{and} \quad S = \sup_n |P_n(\xi)|(a_n^2 + b_n^2 + c_n^2).$$

By (7) we have

$$I = -4\inf_n \max(g_2^2(n), g_3^2(n))/D^2 \quad \text{and} \quad S = -4\sup_n \max(g_2^2(n), g_3^2(n))/D^2$$

One can see that

$$(g_1(n))^2 + (g_2(n))^2 + (g_3(n))^2 = A\sin(n\beta)\cos(n\beta) + B\cos^2(n\beta) + C,$$

where $A = A(\xi,\beta), B = B(\xi,\beta), C = C(\xi,\beta)$. One can show that the function

$$f(x) = Ax\sqrt{1-x^2} + Bx^2 + C$$

attains its minimum(maximum) values at

$$x_{min} = -\frac{1}{\sqrt{2}}\sqrt{1 - \frac{B}{\sqrt{A^2+B^2}}} \quad \text{and} \quad x_{max} = \frac{1}{\sqrt{2}}\sqrt{1 + \frac{B}{\sqrt{A^2+B^2}}}$$

and these values are

$$f_{min} = \frac{B - \sqrt{A^2+B^2}}{2} + C \quad \text{and} \quad f_{max} = \frac{B + \sqrt{A^2+B^2}}{2} + C. \tag{8}$$

Using explicit expressions for $A(\xi,\beta), B(\xi,\beta)$ and $C(\xi,\beta)$ in (8) [**here the algebra becomes really ugly and I don't know how to make it more elegant yet**], we deduce that I and S are roots of the following polynomial:

$$pl = c_6 z^6 + \dots c_2 z^2 + c_1 z + c_0,$$

with

$$c_6 = d^3$$
$$c_5 = 4\,({a_2}^4 - {a_2}^3 - 4\,{a_2}^2 a_1 + {a_2}^2 + 4\,a_2 a_1 + {a_1}^2 - 6\,a_2 - 3\,a_1 + 9)\,d^2$$
$$c_4 = -4\,({a_2}^4 - 4\,{a_2}^2 a_1 + {a_2}^2 + 2\,{a_1}^2 - 8\,a_2 - 2\,a_1 + 3)\,d^2$$
$$c_3 = -16\,({a_2}^3{a_1}^2 + 6\,{a_2}^4 + 6\,{a_2}^2{a_1}^2 - 4\,a_2{a_1}^3 + 2\,{a_1}^4 + 4\,{a_2}^3 - 18\,{a_2}^2 a_1 - 4\,{a_1}^3 + 10\,{a_2}^2$$
$$-10\,a_2 a_1 + 10\,{a_1}^2 - 27\,a_2)\,d$$
$$c_2 = 16\,(2\,{a_2}^4 + 5\,{a_2}^2{a_1}^2 + {a_1}^4 + 2\,{a_2}^3 - 2\,{a_1}^3 + 8\,{a_2}^2 + {a_1}^2 - 6\,a_2)\,d$$
$$c_1 = 64\,{a_2}^2{a_1}^4 + 64\,{a_1}^6 + 320\,{a_2}^3{a_1}^2 + 64\,{a_2}^2{a_1}^3 + 384\,a_2{a_1}^4 - 64\,{a_1}^5 + 320\,{a_2}^4 + 256\,{a_2}^3 a_1$$
$$+704\,{a_2}^2{a_1}^2 - 384\,a_2{a_1}^3 + 128\,{a_1}^4 + 576\,{a_2}^3 - 384\,{a_2}^2 a_1 + 576\,a_2{a_1}^2 + 192\,{a_1}^3 + 1152\,{a_2}^2$$
$$c_0 = -64\,({a_2}^2 + a_2 a_1 + {a_1}^2 + 2\,a_2 - 2\,a_1 + 4)\,({a_2}^2 - a_2 a_1 + {a_1}^2)\,,$$

where a_1 and a_2 are coefficients of $f(x) = x^3 + a_2 x^2 + a_1 x - 1$ and

$$d = -4a_1^3 + 4a_2^3 + a_1^2 a_2^2 - 18 a_1 a_2 - 27$$

is the discriminant of f. For example,

if $f(x) = x^3 + x^2 + x - 1$, then $pl = 64\,(11\,z^3 + 11\,z^2 + z - 1)\,(121\,z^3 - 143\,z^2 + 55\,z - 7)$

if $f(x) = x^3 + x^2 - 1$, then $pl = 12167\,z^6 - 8464\,z^5 - 6348\,z^4 + 2576\,z^3 + 2208\,z^2 - 2048\,z + 448$

if $f(x) = x^3 + x - 1$, then $pl = 29791\,z^6 - 26908\,z^5 + 11532\,z^4 - 3968\,z^3 - 320\,z + 192$

if $f(x) = x^3 - x^2 + 2x - 1$, then $pl = 12167\,z^6 + 19044\,z^4 - 42320\,z^3 + 13984\,z^2 - 2944\,z + 448$

Using explicit expressions for $A(\xi, \beta), B(\xi, \beta)$ and $C(\xi, \beta)$ in (8) [**here the algebra becomes really ugly and I don't know how to make it more elegant yet**], we deduce that I and S are roots of the following polynomial:

$$pl = c_6 z^6 + \dots c_2 z^2 + c_1 z + c_0,$$

with

$$c_6 = d^3$$

$$c_5 = 4\left({a_2}^4 - {a_2}^3 - 4\,{a_2}^2 a_1 + {a_2}^2 + 4\,a_2 a_1 + {a_1}^2 - 6\,a_2 - 3\,a_1 + 9\right) d^2$$

$$c_4 = -4\left({a_2}^4 - 4\,{a_2}^2 a_1 + {a_2}^2 + 2\,{a_1}^2 - 8\,a_2 - 2\,a_1 + 3\right) d^2$$

$$c_3 = -16\left({a_2}^3 {a_1}^2 + 6\,{a_2}^4 + 6\,{a_2}^2 {a_1}^2 - 4\,a_2 {a_1}^3 + 2\,{a_1}^4 + 4\,{a_2}^3 - 18\,{a_2}^2 a_1 - 4\,{a_1}^3 + 10\,{a_2}^2 \right.$$
$$\left. -10\,a_2 a_1 + 10\,{a_1}^2 - 27\,a_2\right) d$$

$$c_2 = 16\left(2\,{a_2}^4 + 5\,{a_2}^2 {a_1}^2 + {a_1}^4 + 2\,{a_2}^3 - 2\,{a_1}^3 + 8\,{a_2}^2 + {a_1}^2 - 6\,a_2\right) d$$

$$c_1 = 64\,{a_2}^2 {a_1}^4 + 64\,{a_1}^6 + 320\,{a_2}^3 {a_1}^2 + 64\,{a_2}^2 {a_1}^3 + 384\,a_2 {a_1}^4 - 64\,{a_1}^5 + 320\,{a_2}^4 + 256\,{a_2}^3 a_1$$
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$$c_0 = -64\left({a_2}^2 + a_2 a_1 + {a_1}^2 + 2\,a_2 - 2\,a_1 + 4\right)\left({a_2}^2 - a_2 a_1 + {a_1}^2\right),$$

where a_1 and a_2 are coefficients of $f(x) = x^3 + a_2 x^2 + a_1 x - 1$ and

$$d = -4a_1^3 + 4a_2^3 + a_1^2 a_2^2 - 18 a_1 a_2 - 27$$

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c_6 &= d^3 \\
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c_4 &= -4\,\left({a_2}^4 - 4\,{a_2}^2 a_1 + {a_2}^2 + 2\,{a_1}^2 - 8\,a_2 - 2\,a_1 + 3\right) d^2 \\
c_3 &= -16\,\left({a_2}^3 {a_1}^2 + 6\,{a_2}^4 + 6\,{a_2}^2 {a_1}^2 - 4\,a_2 {a_1}^3 + 2\,{a_1}^4 + 4\,{a_2}^3 - 18\,{a_2}^2 a_1 - 4\,{a_1}^3 + 10\,{a_2}^2\right. \\
&\quad \left. -10\,a_2 a_1 + 10\,{a_1}^2 - 27\,a_2\right) d \\
c_2 &= 16\,\left(2\,{a_2}^4 + 5\,{a_2}^2 {a_1}^2 + {a_1}^4 + 2\,{a_2}^3 - 2\,{a_1}^3 + 8\,{a_2}^2 + {a_1}^2 - 6\,a_2\right) d \\
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&\quad +704\,{a_2}^2 {a_1}^2 - 384\,a_2 {a_1}^3 + 128\,{a_1}^4 + 576\,{a_2}^3 - 384\,{a_2}^2 a_1 + 576\,a_2 {a_1}^2 + 192\,{a_1}^3 + 1152\,{a_2}^2 \\
c_0 &= -64\,\left({a_2}^2 + a_2 a_1 + {a_1}^2 + 2\,a_2 - 2\,a_1 + 4\right)\left({a_2}^2 - a_2 a_1 + {a_1}^2\right),
\end{aligned}$$

where a_1 and a_2 are coefficients of $f(x) = x^3 + a_2 x^2 + a_1 x - 1$ and

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<u>QUESTION</u>: Describe a_1 as a function of a_2 such that S/I is minimal.

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$$c_0 = -64\,({a_2}^2 + a_2 a_1 + {a_1}^2 + 2\,a_2 - 2\,a_1 + 4)\,({a_2}^2 - a_2 a_1 + {a_1}^2),$$

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$x^3 + x - 1$	0.3458583	0.6276401	1.814732
$x^3 + x^2 + x - 1$	0.2368398	0.5308443	2.241364
$x^3 + 2\,x^2 + 2\,x - 1$	0.1034274	0.5313413	5.137334
$x^3 + 3\,x^2 + 5\,x - 1$	0.02959207	0.3010073	10.17189
$x^3 + 4\,x^2 + 8\,x - 1$	0.01307436	0.2262951	17.30832
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c_6 &= d^3 \\
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c_4 &= -4\,({a_2}^4 - 4\,{a_2}^2 a_1 + {a_2}^2 + 2\,{a_1}^2 - 8\,a_2 - 2\,a_1 + 3)\,d^2 \\
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&\quad -10\,a_2 a_1 + 10\,{a_1}^2 - 27\,a_2)\,d \\
c_2 &= 16\,(2\,{a_2}^4 + 5\,{a_2}^2 {a_1}^2 + {a_1}^4 + 2\,{a_2}^3 - 2\,{a_1}^3 + 8\,{a_2}^2 + {a_1}^2 - 6\,a_2)\,d \\
c_1 &= 64\,{a_2}^2 {a_1}^4 + 64\,{a_1}^6 + 320\,{a_2}^3 {a_1}^2 + 64\,{a_2}^2 {a_1}^3 + 384\,a_2 {a_1}^4 - 64\,{a_1}^5 + 320\,{a_2}^4 + 256\,{a_2}^3 a_1 \\
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\end{aligned}$$

where a_1 and a_2 are coefficients of $f(x) = x^3 + a_2 x^2 + a_1 x - 1$ and

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$x^3 + 10\,x^2 + 51\,x - 1$	0.0003778803	0.03840762	101.6396

<u>ANSWER</u>: We have

$$a_1 = \begin{cases} 2 & \text{if } a_2 = 2 \\ 8 & \text{if } a_2 = 4 \\ \left\lceil \dfrac{a_2^2+1}{2} \right\rceil & \text{otherwise} \end{cases}$$

We now find
$$\inf_n |P_n(\xi)| \max(b_n^2, c_n^2) \quad \text{and} \quad \sup_n |P_n(\xi)| \max(b_n^2, c_n^2).$$
Put
$$I = \inf_n |P_n(\xi)| \max(b_n^2, c_n^2) \quad \text{and} \quad S = \sup_n |P_n(\xi)| \max(b_n^2, c_n^2).$$
By (7) we have
$$I = -4 \inf_n \max(g_2^2(n), g_3^2(n))/D^2 \quad \text{and} \quad S = -4 \sup_n \max(g_2^2(n), g_3^2(n))/D^2 \tag{9}$$
One can show that $\max(g_2^2(n), g_3^2(n))$ attains its infinum value if either $g_2(n) = g_3(n)$ or $g_2(n) = -g_3(n)$, that is when
$$\tan(n\beta) = \pm \frac{\sin\beta(-2\xi^{1/2}\cos\beta + \xi)}{-2\xi^{1/2}\cos^2\beta + \xi\cos\beta + \xi^{7/2} - \xi^{5/2} + \xi^{1/2}}$$
Substituting this into (9) we obtain that I is a root of
$$pl = c_3 z^3 + c_1 z + 1,$$
with
$$c_3 = (a_1a_2 + a_2^2 + a_1 + 2a_2 + 2)^2 d, \quad c_1 = -(a_2^4 - 4a_1a_2^2 + 2a_2^3 + a_1^2 - 7a_1a_2 + a_2^2 - 3a_1 - 6a_2 - 9),$$
or
$$c_3 = (a_1a_2 - a_2^2 - a_1 + 2a_2)^2 d, \quad c_1 = -(a_2^4 - 4a_1a_2^2 - 2a_2^3 + a_1^2 + 7a_1a_2 + a_2^2 - 3a_1 - 6a_2 + 9).$$
Similarly, S is a root of
$$pl = -d^2 z^3 + c_2 z^2 + c_0,$$
with
$$c_2 = -4(a_2^4 - 4a_1a_2^2 + a_1^2 - 6a_2)d, \quad c_0 = 64(a_1a_2 + 1)^2,$$
or
$$c_2 = -4(a_2^2 - 3a_1)d, \quad c_0 = 64,$$
where a_1 and a_2 are coefficients of $f(x) = x^3 + a_2x^2 + a_1x - 1$ and
$$d = -4a_1^3 + 4a_2^3 + a_1^2a_2^2 - 18a_1a_2 - 27$$
is the discriminant of f. For example,

if $f(x) = x^3 + x^2 + x - 1$, then inf $= 0.1221..$ is a root of $pl = 2156z^3 - 24z - 1$ (first) and sup $= 0.3503..$ is a root of $pl = -121z^3 - 88z^2 + 16$ (first)

if $f(x) = x^3 + x^2 - 1$, then inf $= 0.1711..$ is a root of $pl = 575z^3 - 11z - 1$ (first) and sup $= 0.5598..$ is a root of $pl = -529z^3 + 92z^2 + 64$ (second)

if $f(x) = x^3 + x - 1$, then inf $= 0.1325..$ is a root of $pl = 31z^3 + 7z - 1$ (second) and sup $= 0.4532..$ is a root of $pl = -961z^3 + 124z^2 + 64$ (first)

Consider a polynomial $f(x) = x^3 + a_2x^2 + a_1x - 1$ in $\mathbb{Z}[x]$ irreducible over $\mathbb{Q}$. Let ξ, ξ_1 and ξ_2 be its roots. Assume that ξ is real and ξ_1, ξ_2 are complex. We also assume that $|\xi| < 1$.

We introduce a sequence of polynomials $P_n(x) = a_n + b_nx + c_nx^2$ in $\mathbb{Z}[x]$, defined by $P_n(x) = x^n \mod f$. Since $f(\xi) = f(\xi_1) = f(\xi_2) = 0$, we have

$$
\begin{aligned}
P_n(\xi) &= \xi^n = a_n + b_n\xi + c_n\xi^2, \\
P_n(\xi_1) &= \xi_1^n = a_n + b_n\xi_1 + c_n\xi_1^2, \\
P_n(\xi_2) &= \xi_2^n = a_n + b_n\xi_2 + c_n\xi_2^2.
\end{aligned} \tag{1}
$$

In other words,

$$
\begin{pmatrix} \xi^n \\ \xi_1^n \\ \xi_2^n \end{pmatrix} = V \begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix}, \quad \text{where} \quad V = \begin{pmatrix} 1 & \xi & \xi^2 \\ 1 & \xi_1 & \xi_1^2 \\ 1 & \xi_2 & \xi_2^2 \end{pmatrix} \tag{2}
$$

is the Vandermonde matrix. From (2) it follows that

$$
\begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} = V^{-1} \begin{pmatrix} \xi^n \\ \xi_1^n \\ \xi_2^n \end{pmatrix}. \tag{3}
$$

Since ξ, ξ_1 and ξ_2 are roots of $f(x)$ and the last coefficient of f is -1, it follows that $|\xi\xi_1\xi_2| = 1$. Therefore

$$
|\xi|^{-1/2} = |\xi_1| = |\xi_2|, \tag{4}
$$

because ξ_1 and ξ_2 are complex conjugates. Applying (4) to (3), we get

$$
\begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} = V^{-1}\xi^{-n/2} \begin{pmatrix} \xi^{3n/2} \\ \cos(n\beta) + i\sin(n\beta) \\ \cos(n\beta) - i\sin(n\beta) \end{pmatrix}, \tag{5}
$$

where $\beta = \arg(\xi_1)$. Observe that for large n, $\xi^{3n/2}$ is much smaller than $|\cos(n\beta) \pm i\sin(n\beta)|$, therefore

$$
\begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} \approx V^{-1}\xi^{-n/2} \begin{pmatrix} 0 \\ \cos(n\beta) + i\sin(n\beta) \\ \cos(n\beta) - i\sin(n\beta) \end{pmatrix},
$$

which can be rewritten as

$$
\xi^{n/2} \begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} \approx \frac{2i}{D} \begin{pmatrix} \xi^{3/2}\sin((n-1)\beta) - \sin((n-2)\beta) \\ \xi^{-1}\sin((n-2)\beta) - \xi^2\sin(n\beta) \\ \xi\sin(n\beta) - \xi^{-1/2}\sin((n-1)\beta) \end{pmatrix}, \tag{6}
$$

where D is the determinant of V.

Consider a polynomial $f(x) = 2x^3 - 2x^2 - 2x + 1$. It is irreducible over $\mathbb{Q}$ and all its roots $\xi = 0.4030..$, $\xi_1 = -0.8546..$ and $\xi_2 = 1.4516..$ are real.

We introduce a sequence of polynomials $P_n(x) = a_n + b_n x + c_n x^2$ in $\mathbb{Z}[x]$, defined by

$$P_n(x) = 2^k (x^2 + 12x - 5)^n \mod f,$$

were

$$k = \begin{cases} 2\lfloor n/3 \rfloor & \text{if } n \equiv 0 \text{ or } 1 \mod 3 \\ 2\lfloor n/3 \rfloor + 1 & \text{otherwise.} \end{cases}$$

One can show that P_n have integer coefficients. Since $f(\xi) = f(\xi_1) = f(\xi_2) = 0$, we have

$$P_n(\xi) = 2^k(\xi^2 + 12\xi - 5)^n = a_n + b_n\xi + c_n\xi^2,$$

$$P_n(\xi_1) = 2^k(\xi_1^2 + 12\xi_1 - 5)^n = a_n + b_n\xi_1 + c_n\xi_1^2,$$

$$P_n(\xi_2) = 2^k(\xi_2^2 + 12\xi_2 - 5)^n = a_n + b_n\xi_2 + c_n\xi_2^2.$$

In other words,

$$2^k \begin{pmatrix} (\xi^2 + 12\xi - 5)^n \\ (\xi_1^2 + 12\xi_1 - 5)^n \\ (\xi_2^2 + 12\xi_2 - 5)^n \end{pmatrix} = V \begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix}, \quad \text{where} \quad V = \begin{pmatrix} 1 & \xi & \xi^2 \\ 1 & \xi_1 & \xi_1^2 \\ 1 & \xi_2 & \xi_2^2 \end{pmatrix}$$

is the Vandermonde matrix. From this it follows that

$$\begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} = 2^k\, V^{-1} \begin{pmatrix} (\xi^2 + 12\xi - 5)^n \\ (\xi_1^2 + 12\xi_1 - 5)^n \\ (\xi_2^2 + 12\xi_2 - 5)^n \end{pmatrix},$$

so

$$2^{k/2}(\xi^2 + 12\xi - 5)^{n/2} \begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} = 2^{3k/2}\, V^{-1} \begin{pmatrix} (\xi^2 + 12\xi - 5)^{3n/2} \\ (\xi_1^2 + 12\xi_1 - 5)^n (\xi^2 + 12\xi - 5)^{n/2} \\ (\xi_2^2 + 12\xi_2 - 5)^n (\xi^2 + 12\xi - 5)^{n/2} \end{pmatrix}$$

$$\approx V^{-1} \begin{pmatrix} 0 \\ [2(\xi_1^2 + 12\xi_1 - 5)(\xi^2 + 12\xi - 5)^{1/2}]^n \\ [2(\xi_2^2 + 12\xi_2 - 5)(\xi^2 + 12\xi - 5)^{1/2}]^n \end{pmatrix}.$$

A straightforward calculation shows that $2(\xi_1^2 + 12\xi_1 - 5)(\xi^2 + 12\xi - 5)^{1/2} \approx -.9999592172$ and $2(\xi_2^2 + 12\xi_2 - 5)(\xi^2 + 12\xi - 5)^{1/2} \approx 1.000040785$. Note that $1.000040785^{10000} \approx 1.5$. In other words, the drift of the scaled errors will be almost invisible for at least 10000 iterations.

Consider a polynomial $f(x) = x^3 + a_2x^2 + a_1x + 1$ in $\mathbb{Z}[x]$ irreducible over $\mathbb{Q}$ such that $a_1 + a_2 = -3$. Let ξ, ξ_1 and ξ_2 be its roots. Assume that all three roots are real and that the fundamental units are x and $x-1$. We also assume that $0 < \xi < 1, \xi_1 < 0, \xi_2 > 1$. Then $g(x) = x^3 + (a_2+3)x^2 + (a_1 + 2a_2 + 3)x - 1$ is a minimal polynomial of $\xi - 1$. Let γ be such a number that

$$\frac{|\xi_1|}{\xi_2} = \left(\frac{\xi_2 - 1}{1 - \xi_1}\right)^{\gamma}. \tag{1}$$

Put

$$m = \lfloor n\gamma \rfloor, \quad n \in \mathbb{Z}^+. \tag{2}$$

We introduce a sequence of polynomials $P_n(x) = a_n + b_nx + c_nx^2$ in $\mathbb{Z}[x]$, defined by $P_n(x) = x^n(x-1)^m \bmod f$. Since $f(\xi) = f(\xi_1) = f(\xi_2) = 0$, we have

$$\begin{aligned} P_n(\xi) &= \xi^n(\xi - 1)^m = a_n + b_n\xi + c_n\xi^2, \\ P_n(\xi_1) &= \xi_1^n(\xi_1 - 1)^m = a_n + b_n\xi_1 + c_n\xi_1^2, \\ P_n(\xi_2) &= \xi_2^n(\xi_2 - 1)^m = a_n + b_n\xi_2 + c_n\xi_2^2. \end{aligned} \tag{3}$$

In other words,

$$\begin{pmatrix} \xi^n(\xi-1)^m \\ \xi_1^n(\xi_1-1)^m \\ \xi_2^n(\xi_2-1)^m \end{pmatrix} = V \begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix}, \quad \text{where} \quad V = \begin{pmatrix} 1 & \xi & \xi^2 \\ 1 & \xi_1 & \xi_1^2 \\ 1 & \xi_2 & \xi_2^2 \end{pmatrix} \tag{4}$$

is the Vandermonde matrix. From (4) it follows that

$$\begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} = V^{-1} \begin{pmatrix} \xi^n(\xi-1)^m \\ \xi_1^n(\xi_1-1)^m \\ \xi_2^n(\xi_2-1)^m \end{pmatrix},$$

so

$$\xi^{n/2}(1-\xi)^{m/2} \begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} = V^{-1} \begin{pmatrix} (-1)^m\xi^{3n/2}(1-\xi)^{3m/2} \\ (-1)^m(\xi^{1/2}\xi_1)^n((1-\xi)^{1/2}(1-\xi_1))^m \\ (\xi^{1/2}\xi_2)^n((1-\xi)^{1/2}(\xi_2-1))^m \end{pmatrix}. \tag{5}$$

Since ξ, ξ_1 and ξ_2 are roots of $f(x)$ and the last coefficient of f is 1, it follows that $|\xi\xi_1\xi_2| = 1$. Similarly, since $\xi - 1, \xi_1 - 1$ and $\xi_2 - 1$ are roots of $g(x)$ and the last coefficient of g is -1, it follows that $|(\xi-1)(\xi_1 - 1)(\xi_2 - 1)| = 1$. Also, for a large m the first entry is much smaller than the last two. From this, (1) and (2) it follows that

$$\xi^{n/2}(1-\xi)^{m/2} \begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} \approx (-1)^m V^{-1} \begin{pmatrix} 0 \\ \nu \\ \nu^{-1} \end{pmatrix}, \quad \text{where} \quad \nu = \left(\frac{\xi_2 - 1}{1-\xi_1}\right)^{\frac{n\gamma - m}{2}}.$$

Since

$$V^{-1} = \frac{1}{D} \begin{pmatrix} \xi_1\xi_2(\xi_2 - \xi_1) & \xi\xi_2(\xi - \xi_2) & \xi\xi_1(\xi_1 - \xi) \\ \xi_1^2 - \xi_2^2 & \xi_2^2 - \xi^2 & \xi^2 - \xi_1^2 \\ \xi_2 - \xi_1 & \xi - \xi_2 & \xi_1 - \xi \end{pmatrix},$$

where D is the determinant of V, we have

$$\xi^{n/2}(1-\xi)^{m/2}\begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} \approx \frac{(-1)^m}{D}\begin{pmatrix} \xi\xi_2(\xi-\xi_2)\nu - \xi\xi_1(\xi-\xi_1)\nu^{-1} \\ (\xi_2^2-\xi^2)\nu + (\xi^2-\xi_1^2)\nu^{-1} \\ (\xi-\xi_2)\nu + (\xi_1-\xi)\nu^{-1} \end{pmatrix}. \tag{6}$$

Put

$$g_1(n) = \xi\xi_2(\xi-\xi_2)\nu - \xi\xi_1(\xi-\xi_1)\nu^{-1},$$

$$g_2(n) = (\xi_2^2-\xi^2)\nu + (\xi^2-\xi_1^2)\nu^{-1},$$

$$g_3(n) = (\xi-\xi_2)\nu + (\xi_1-\xi)\nu^{-1}.$$

From (3) and (6) it follows that

$$(P_n(\xi))^{1/2}a_n \approx \frac{(-1)^m}{D}g_1(n),$$

$$(P_n(\xi))^{1/2}b_n \approx \frac{(-1)^m}{D}g_2(n), \tag{7}$$

$$(P_n(\xi))^{1/2}c_n \approx \frac{(-1)^m}{D}g_3(n).$$

To show that the points $(g_1(n), g_2(n), g_3(n))$ trace a hyperbola in 3-dimensional space, we note that

$$g_1(n) + \xi g_2(n) + \xi^2 g_3(n) = 0$$

and

$$-\frac{g_1^2(n)}{a^2} + \frac{g_2^2(n)}{b^2} + \frac{g_3^2(n)}{c^2} = 1,$$

where

$$a^2 = \frac{-2D\xi^3(\xi_2-\xi_1)}{2\xi+\xi_1+\xi_2}, \quad b^2 = \frac{-2D\xi(\xi_2-\xi_1)}{\xi_1+\xi_2}, \quad c^2 = \frac{2D(\xi_2-\xi_1)}{\xi\xi_1+\xi\xi_2+2\xi_1\xi_2}.$$