On approximation of real, complex, and p-adic numbers by algebraic numbers of bounded degree

BY

K. I. Tsishchanka

I. On approximation by rational numbers

THEOREM 1 **(Dirichlet, 1842). For any real irrational number ξ there exist infinitely many rational numbers p/q such that**

$$\left|\xi - \frac{p}{q}\right| < \frac{1}{q^2}.$$

Example: Let $\xi = e$. Consider the continued fraction expansion:

$$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \dots}}}}$$

We have

$$2 + \frac{1}{1} = 3 \qquad 2 + \cfrac{1}{1 + \cfrac{1}{2}} = \frac{8}{3} \qquad 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1}}} = \frac{11}{4}$$

and so on.

The first convergents are:

$$\delta_1 = 3 \qquad |e - 3| < 1$$

$$\delta_2 = \frac{8}{3} \qquad \left|e - \frac{8}{3}\right| < \frac{1}{3^2}$$

$$\delta_3 = \frac{11}{4} \qquad \left|e - \frac{11}{4}\right| < \frac{1}{4^2}$$

$$\delta_4 = \frac{19}{7} \qquad \left|e - \frac{19}{7}\right| < \frac{1}{7^2}$$

$$\delta_5 = \frac{87}{32} \qquad \left|e - \frac{87}{32}\right| < \frac{1}{32^2}$$

$$\delta_6 = \frac{106}{39} \qquad \left|e - \frac{106}{39}\right| < \frac{1}{39^2}$$

We also note that

$$\begin{aligned}
\delta_1 &= 3 & \quad |e - 3| &< \frac{1}{2 \cdot 1^2} \\
\delta_2 &= \frac{8}{3} & \quad \left|e - \frac{8}{3}\right| &< \frac{1}{2 \cdot 3^2} \\
\delta_3 &= \frac{11}{4} & \quad \left|e - \frac{11}{4}\right| &< \frac{1}{4^2} \\
\delta_4 &= \frac{19}{7} & \quad \left|e - \frac{19}{7}\right| &< \frac{1}{2 \cdot 7^2} \\
\delta_5 &= \frac{87}{32} & \quad \left|e - \frac{87}{32}\right| &< \frac{1}{2 \cdot 32^2} \\
\delta_6 &= \frac{106}{39} & \quad \left|e - \frac{106}{39}\right| &< \frac{1}{39^2}
\end{aligned}$$

THEOREM 2. **For any real irrational number ξ there exist infinitely many rational numbers p/q such that**

$$\left|\xi - \frac{p}{q}\right| < \frac{1}{2q^2}.$$

<u>THEOREM 2</u>. **For any real irrational number $\boldsymbol{\xi}$ there exist infinitely many rational numbers $\boldsymbol{p/q}$ such that**

$$\left|\xi - \frac{p}{q}\right| < \frac{1}{2q^2}.$$

<u>THEOREM 1.</u> **For any real irrational number $\boldsymbol{\xi}$ there exist infinitely many rational numbers $\boldsymbol{p/q}$ such that**

$$\left|\xi - \frac{p}{q}\right| < \frac{1}{q^2}.$$

Finally, some convergents give even better approximation:

$$\boldsymbol{\delta_1 = 3 \qquad |e - 3| < \frac{1}{\sqrt{5} \cdot 1^2}}$$

$$\boldsymbol{\delta_2 = \frac{8}{3} \qquad \left|e - \frac{8}{3}\right| < \frac{1}{2 \cdot 3^2}}$$

$$\boldsymbol{\delta_3 = \frac{11}{4} \qquad \left|e - \frac{11}{4}\right| < \frac{1}{4^2}}$$

$$\boldsymbol{\delta_4 = \frac{19}{7} \qquad \left|e - \frac{19}{7}\right| < \frac{1}{\sqrt{5} \cdot 7^2}}$$

$$\boldsymbol{\delta_5 = \frac{87}{32} \qquad \left|e - \frac{87}{32}\right| < \frac{1}{2 \cdot 32^2}}$$

$$\boldsymbol{\delta_6 = \frac{106}{39} \qquad \left|e - \frac{106}{39}\right| < \frac{1}{39^2}}$$

<u>THEOREM 3</u> **(Hurwitz). For any real irrational number $\boldsymbol{\xi}$ there exist infinitely many rational numbers $\boldsymbol{p/q}$ such that**

$$\left|\xi - \frac{p}{q}\right| < \frac{1}{\sqrt{5}q^2}.$$

THEOREM 3 (Hurwitz). For any real irrational number ξ there exist infinitely many rational numbers p/q such that

$$\left|\xi - \frac{p}{q}\right| < \frac{1}{\sqrt{5}q^2}.$$

This result is the best possible.

Theorem 3 (Hurwitz). **For any real irrational number ξ there exist infinitely many rational numbers p/q such that**

$$\left|\xi - \frac{p}{q}\right| < \frac{1}{\sqrt{5}q^2}.$$

This result is the best possible.

Example:

$$\xi = \frac{1+\sqrt{5}}{2} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \dots}}}}$$

II. Polynomial Interpretation

THEOREM 1. **For any real irrational number ξ there exist infinitely many rational numbers p/q such that**

$$\left|\xi - \frac{p}{q}\right| < \frac{1}{q^2}.$$

II. Polynomial Interpretation

<u>THEOREM 1.</u> **For any real irrational number ξ there exist infinitely many rational numbers p/q such that**

$$\left|\xi - \frac{p}{q}\right| < \frac{1}{q^2} \quad \Longrightarrow \quad |q\xi - p| < q^{-1}.$$

II. Polynomial Interpretation

<u>THEOREM 1.</u> **For any real irrational number ξ there exist infinitely many rational numbers p/q such that**

$$\left|\xi - \frac{p}{q}\right| < \frac{1}{q^2} \quad \Longrightarrow \quad |q\xi - p| < q^{-1}.$$

<u>THEOREM 4</u>. **For any real irrational number ξ there exist infinitely many polynomials $P \in Z[x]$ of the first degree such that**

$$|P(\xi)| \ll \lceil P \rceil^{-1},$$

where $\lceil P \rceil$ denotes the height of P.

II. Polynomial Interpretation

<u>Theorem 1.</u> **For any real irrational number ξ there exist infinitely many rational numbers p/q such that**

$$\left|\xi - \frac{p}{q}\right| < \frac{1}{q^2} \quad \Longrightarrow \quad |q\xi - p| < q^{-1}.$$

<u>Theorem 5.</u> **For any real number $\xi \notin A_n$ there exist infinitely many polynomials $P \in Z[x]$ of degree $\leq n$ such that**

$$|P(\xi)| \ll \lceil P \rceil^{-n},$$

where A_n is the set of real algebraic numbers of degree $\leq n$.

$$\left|\xi - \frac{p}{q}\right| < q^{-2}$$

$$\left|\boldsymbol{\xi} - \frac{\boldsymbol{p}}{\boldsymbol{q}}\right| < \boldsymbol{q}^{-2} \longrightarrow |\boldsymbol{q}\boldsymbol{\xi} - \boldsymbol{p}| < \boldsymbol{q}^{-1}$$

$$\left|\xi - \frac{p}{q}\right| < q^{-2} \quad \longrightarrow \quad |q\xi - p| < q^{-1}$$

$$\downarrow$$

$$|P(\xi)| \ll \lceil P \rceil^{-n}$$

$$
\begin{array}{ccc}
\left|\xi - \dfrac{p}{q}\right| < q^{-2} & \longrightarrow & |q\xi - p| < q^{-1} \\
 & & \downarrow \\
|\xi - \alpha| & \longleftarrow & |P(\xi)| \ll \overline{|P|}^{-n}
\end{array}
$$

$$
\begin{array}{ccc}
\left|\xi - \dfrac{p}{q}\right| < q^{-2} & \longrightarrow & |q\xi - p| < q^{-1} \\
 & & \downarrow \\
|\xi - \alpha| \ll \, ? & \longleftarrow & |P(\xi)| \ll \lceil P \rceil^{-n}
\end{array}
$$

III. Conjecture of Wirsing

<u>Conjecture</u> (Wirsing, 1961). **For any real number $\xi \notin A_n$, there exist infinitely many algebraic numbers $\alpha \in A_n$ with**

$$|\xi - \alpha| \ll H(\alpha)^{-n-1+\epsilon}, \quad \epsilon > 0,$$

where $H(\alpha)$ is the height of α.

Further W. M. Schmidt conjectured that the exponent

$$-n - 1 + \epsilon$$

can be replaced even by

$$-n - 1.$$

$$\begin{array}{ccc} \left|\boldsymbol{\xi}-\dfrac{\boldsymbol{p}}{\boldsymbol{q}}\right|<\boldsymbol{q}^{-2} & \longrightarrow & |\boldsymbol{q}\boldsymbol{\xi}-\boldsymbol{p}|<\boldsymbol{q}^{-1} \\ & & \downarrow \\ |\boldsymbol{\xi}-\boldsymbol{\alpha}| \ll \boldsymbol{H}(\boldsymbol{\alpha})^{-n-1} & \longleftarrow & |\boldsymbol{P}(\boldsymbol{\xi})| \ll \lceil \boldsymbol{P} \rceil^{-n} \end{array}$$

$$
\begin{array}{ccc}
\left|\xi - \dfrac{p}{q}\right| < q^{-2} & \longrightarrow & |q\xi - p| < q^{-1} \\
 & & \downarrow \\
|\xi - \alpha| \ll H(\alpha)^{-n-1} & \longleftarrow & |P(\xi)| \ll \lceil P \rceil^{-n}
\end{array}
$$

At the moment this Conjecture is proved only for

$$n = 1 \;\Rightarrow\; |\xi - \alpha| \ll H(\alpha)^{-2} \text{ (Dirichlet)}$$

$$
\begin{array}{ccc}
\left|\xi - \dfrac{p}{q}\right| < q^{-2} & \longrightarrow & |q\xi - p| < q^{-1} \\
 & & \downarrow \\
|\xi - \alpha| \ll H(\alpha)^{-n-1} & \longleftarrow & |P(\xi)| \ll \lceil P \rceil^{-n}
\end{array}
$$

At the moment this Conjecture is proved only for

$n = 1 \;\Rightarrow\; |\xi - \alpha| \ll H(\alpha)^{-2}$ **(Dirichlet)**

$n = 2 \;\Rightarrow\; |\xi - \alpha| \ll H(\alpha)^{-3}$ **(Davenport – Schmidt)**

$$\begin{array}{ccc}
\left|\xi - \dfrac{p}{q}\right| < q^{-2} & \longrightarrow & |q\xi - p| < q^{-1} \\
 & & \downarrow \\
|\xi - \alpha| \ll H(\alpha)^{-n-1} & \longleftarrow & |P(\xi)| \ll \lceil P \rceil^{-n}
\end{array}$$

At the moment this Conjecture is proved only for

$n = 1 \;\Rightarrow\; |\xi - \alpha| \ll H(\alpha)^{-2}$ **(Dirichlet)**

$n = 2 \;\Rightarrow\; |\xi - \alpha| \ll H(\alpha)^{-3}$ **(Davenport – Schmidt)**

$n > 2 \;\Rightarrow$ **???**

Consider the polynomial

$$P(x) = a_n x^n + \ldots + a_1 x + a_0$$
$$= a_n (x - \alpha_1) \cdot \ldots \cdot (x - \alpha_n).$$

Without loss of generality we can assume that α_1 is the root of $P(x)$ closest to ξ.

It is known that

$$|\xi - \alpha_1| \ll \frac{|P(\xi)|}{|P'(\xi)|}.$$

By Theorem 3 there are infinitely many polynomials $P \in Z[x]$ of degree $\leq n$ such that

$$|P(\xi)| \ll \overline{|P|}^{-n}.$$

Let $n = 1$. Then

$$|P'(\xi)| = |a_1| \asymp \overline{|P|} \quad \Rightarrow \quad \boxed{|\xi - \alpha_1| \ll \frac{\overline{|P|}^{-1}}{\overline{|P|}} = \overline{|P|}^{-2}}$$

It is known that

$$|\xi - \alpha_1| \ll \frac{|P(\xi)|}{|P'(\xi)|}.$$

By Theorem 3 there are infinitely many polynomials $P \in Z[x]$ of degree $\leq n$ such that

$$|P(\xi)| \ll \lceil P \rceil^{-n}.$$

Let $n = 2$. Then for some $\delta \leq 1$ we have

$$|P'(\xi)| = |2a_2\xi + a_1| \asymp \lceil P \rceil^{\delta} \Rightarrow \boxed{|\xi - \alpha_1| \ll \frac{\lceil P \rceil^{-2}}{\lceil P \rceil^{\delta}} = \lceil P \rceil^{-2-\delta}}$$

QUESTION: **Can one prove that the following is impossible: All polynomials with $|P(\xi)| \ll \lceil P \rceil^{-n}$ have a "small" derivative $|P'(\xi)| \asymp \lceil P \rceil^{\delta}$.**

QUESTION: Can one prove that the following is impossible: All polynomials with $|P(\xi)| \ll \lceil P \rceil^{-n}$ have a "small" derivative $|P'(\xi)| \asymp \lceil P \rceil^{\delta}$.

ANSWER: Yes, for n=1 (Dirichlet, 1842)
n=2 (Davenport – Schmidt, 1967)

IV. Theorem of Wirsing

THEOREM 6 (Wirsing, 1961). **For any real number $\xi \notin A_n$ there exist infinitely many algebraic numbers $\alpha \in A_n$ with**

$$|\xi - \alpha| \ll H(\alpha)^{-\frac{n}{2}-\lambda_n}, \quad \lim_{n\to\infty} \lambda_n = 2.$$

By Dirichlet's Box principle there are infinitely many polynomials

$$P(x) = a_n(x - \alpha_1) \cdot \ldots \cdot (x - \alpha_n)$$

such that $|P(\xi)| \ll \boxed{P}^{-n}$, therefore

$$|\xi - \alpha_1| \cdot \ldots \cdot |\xi - \alpha_n| \ll \boxed{P}^{-n} a_n^{-1}.$$

Even if $a_n = \boxed{P}$, we can only prove that

$$|\xi - \alpha_1| \cdot \ldots \cdot |\xi - \alpha_n| \ll \boxed{P}^{-n-1} \ll H(\alpha_1)^{-n-1},$$

$$\Downarrow$$

$$\boxed{|\xi - \alpha_1| \ll H(\alpha_1)^{-\frac{n+1}{n}} \text{ ???}}$$

It is also clear, that the worth case for us is when

$$|\xi - \alpha_1| = \ldots = |\xi - \alpha_n|.$$

QUESTION: **Can one prove that for infinitely many polynomials $P \in Z[x]$ with $|P(\xi)| \ll \lceil P \rceil^{-n}$ the situation**

$$|\xi - \alpha_1| = \ldots = |\xi - \alpha_n|$$

is impossible?

QUESTION: Can one prove that for infinitely many polynomials $P \in Z[x]$ with $|P(\xi)| \ll \overline{P}^{-n}$ the situation

$$|\xi - \alpha_1| = \ldots = |\xi - \alpha_n|$$

is impossible?

ANSWER: For infinitely many polynomials $P \in Z[x]$ with $|P(\xi)| \ll \overline{P}^{-n}$ we have:

$$|\xi - \alpha_1| \ll |\xi - \alpha_2| \ll 1,$$

$|\xi - \alpha_3|, \ldots, |\xi - \alpha_n|$ are "big".

Step 1: Construct infinitly many $P,\ Q \in Z[x]$, deg $P, Q \leq n$, such that

$$|P(\xi)| \ll \boxed{P}^{-n}$$
$$|Q(\xi)| \ll \boxed{Q}^{-n}$$
$$\boxed{P} \ll \boxed{Q}$$

and

P, Q have no common root

Step 2. Consider the resultant of P, Q :

$$\boldsymbol{R(P,Q) = a_m^{\ell} b_{\ell}^{m} \prod_{1 \leq i,j \leq n} (\alpha_i - \beta_j).}$$

Step 2. Consider the resultant of P, Q :

$$R(P,Q) = a_m^{\ell} b_{\ell}^{m} \prod_{1 \leq i,j \leq n} (\alpha_i - \beta_j).$$

On the one hand,

$$R(P,Q) \neq 0,$$

since P, Q have no common root.

Step 2. Consider the resultant of P, Q :

$$R(P,Q) = a_m^{\ell} b_{\ell}^{m} \prod_{1 \le i,j \le n} (\alpha_i - \beta_j).$$

On the one hand,

$$R(P,Q) \neq 0,$$

since P, Q have no common root. Moreover,

$$R(P,Q) \in Z,$$

since P, Q have integer coefficients.

Step 2. Consider the resultant of P, Q :

$$R(P,Q) = a_m^{\ell} b_{\ell}^{m} \prod_{1 \le i,j \le n} (\alpha_i - \beta_j).$$

On the one hand,

$$R(P,Q) \neq 0,$$

since P, Q have no common root. Moreover,

$$R(P,Q) \in Z,$$

since P, Q have integer coefficients. Therefore we get

$$|R(P,Q)| \geq 1.$$

Step 3. On the other hand,

$$|R(P,Q)| = a_m^{\ell} b_{\ell}^{m} \prod_{1\le i,j\le n} |\alpha_i - \beta_j|.$$

Step 3. On the other hand,

$$\begin{aligned}|R(P,Q)| &= a_m^{\ell} b_\ell^{m} \prod_{1\le i,j\le n} |\alpha_i - \beta_j| \\ &\ll \boxed{P}^{2n} \prod_{1\le i,j\le n} |\alpha_i - \beta_j|.\end{aligned}$$

Step 3. On the other hand,

$$
\begin{aligned}
|R(P,Q)| &= a_m^{\ell} b_{\ell}^{m} \prod_{1\le i,j\le n} |\alpha_i - \beta_j| \\
&\ll \lceil P \rceil^{2n} \prod_{1\le i,j\le n} |\alpha_i - \beta_j| \\
&\ll \lceil P \rceil^{2n} \prod_{1\le i,j\le n} \max(|\xi - \alpha_i|,\ |\xi - \beta_j|).
\end{aligned}
$$

Step 3. On the other hand,

$$
\begin{aligned}
|R(P,Q)| &= a_m^{\ell} b_{\ell}^{m} \prod_{1\le i,j\le n} |\alpha_i - \beta_j| \\
&\ll \boxed{P}^{2n} \prod_{1\le i,j\le n} |\alpha_i - \beta_j| \\
&\ll \boxed{P}^{2n} \prod_{1\le i,j\le n} \max(|\xi - \alpha_i|,\ |\xi - \beta_j|).
\end{aligned}
$$

If

$$|\xi - \alpha_1| = \ldots = |\xi - \alpha_n| \ll \boxed{P}^{-1-\frac{1}{n}},$$

$$|\xi - \beta_1| = \ldots = |\xi - \beta_n| \ll \boxed{P}^{-1-\frac{1}{n}}$$

Step 3. On the other hand,

$$
\begin{aligned}
|R(P,Q)| &= a_m^{\ell} b_{\ell}^{m} \prod_{1\le i,j\le n} |\alpha_i - \beta_j| \\
&\ll \lceil P \rceil^{2n} \prod_{1\le i,j\le n} |\alpha_i - \beta_j| \\
&\ll \lceil P \rceil^{2n} \prod_{1\le i,j\le n} \max(|\xi - \alpha_i|,\ |\xi - \beta_j|).
\end{aligned}
$$

If

$$
|\xi - \alpha_1| = \ldots = |\xi - \alpha_n| \ll \lceil P \rceil^{-1-\frac{1}{n}},
$$

$$
|\xi - \beta_1| = \ldots = |\xi - \beta_n| \ll \lceil P \rceil^{-1-\frac{1}{n}},
$$

then

$$
|R(P,Q)| \ll \lceil P \rceil^{2n} \lceil P \rceil^{\left(-1-\frac{1}{n}\right)n^2} = \lceil P \rceil^{n-n^2} < 1
$$

Step 3. On the other hand,

$$
\begin{aligned}
|R(P,Q)| &= a_m^{\ell} b_{\ell}^{m} \prod_{1\le i,j\le n} |\alpha_i - \beta_j| \\
&\ll \boxed{P}^{2n} \prod_{1\le i,j\le n} |\alpha_i - \beta_j| \\
&\ll \boxed{P}^{2n} \prod_{1\le i,j\le n} \max(|\xi - \alpha_i|,\ |\xi - \beta_j|).
\end{aligned}
$$

If

$$|\xi - \alpha_1| = \ldots = |\xi - \alpha_n| \ll \boxed{P}^{-1-\frac{1}{n}},$$

$$|\xi - \beta_1| = \ldots = |\xi - \beta_n| \ll \boxed{P}^{-1-\frac{1}{n}},$$

then

$$|R(P,Q)| \ll \boxed{P}^{2n} \boxed{P}^{\left(-1-\frac{1}{n}\right)n^2} = \boxed{P}^{n-n^2} < 1,$$

which gives a contradiction, since $|R(P,Q)| \geq 1$ by Step 2.

Lemma (Wirsing, 1961):

$$|\xi - \gamma| \ll \max \begin{cases} |P(\xi)|^{\frac{1}{2}} |Q(\xi)| \lceil P \rceil^{n-\frac{3}{2}}, \\ \\ |P(\xi)| |Q(\xi)|^{\frac{1}{2}} \lceil P \rceil^{n-\frac{3}{2}}, \end{cases}$$

where γ is a root of P or Q closest to ξ.
Since

$$|P(\xi)| \ll \lceil P \rceil^{-n}, \quad |Q(\xi)| \ll \lceil Q \rceil^{-n},$$

we get

$$|\xi - \gamma| \ll \lceil P \rceil^{-\frac{n}{2}-n+n-\frac{3}{2}} = \lceil P \rceil^{-\frac{n}{2}-\frac{3}{2}}.$$

V. "Big Derivative" Method

THEOREM 7 (Bernik-Tsishchanka, 1993). **For any real number $\xi \notin A_n$ there exist infinitely many algebraic numbers $\alpha \in A_n$ with**

$$|\xi - \alpha| \ll H(\alpha)^{-\frac{n}{2} - \lambda_n}, \quad \lim_{n\to\infty} \lambda_n = 3.$$

The following table contains the values of

$$\frac{n}{2} + \lambda_n$$

corresponding to Wirsing's Theorem (1961), the Theorem of Bernik-Tsishchanka (1993), and the Conjecture:

n	1961	1993	Conj.
3	3.28	3.5	4
4	3.82	4.12	5
5	4.35	4.71	6
10	6.92	7.47	11
50	26.98	27.84	51
100	51.99	52.92	101

Fix some $H > 0$. By Dirichlet's Box Principle there exists an integer polynomial P such that

$$|a_n| \ll H,\ \ldots,\ |a_2| \ll H,$$
$$|a_1| \ll H^{1+\epsilon}, \quad |a_0| \ll H^{1+\epsilon}, \tag{1}$$
$$|P(\xi)| \ll H^{-n-\epsilon},$$

where $\epsilon > 0$.

Case A: **Let**

$$\max(|a_1|,\ |a_0|) \gg H,$$

that is

$$\max(|a_1|,\ |a_0|) = H^{1+\delta} = \lceil P \rceil, \qquad 0 < \delta \le \epsilon.$$

It is clear that in this case the derivative of P is "big", that is

$$|P'(\xi)| \asymp H^{1+\delta}. \tag{2}$$

We have the following well-known inequality

$$|\xi - \alpha| \ll \frac{|P(\xi)|}{|P'(\xi)|}, \tag{3}$$

where α is the root of the polynomial P closest to ξ. Substituting (1) and (2) into (3), we get

$$|\xi - \alpha| \ll \frac{H^{-n-\epsilon}}{H^{1+\delta}} = H^{-(1+\delta)\frac{n+1+\epsilon+\delta}{1+\delta}} = \lceil P \rceil^{-\frac{n+1+\epsilon+\delta}{1+\delta}} \ll H(\alpha$$

Case B: **Let**

$$\max(|a_1|,\ |a_0|) \ll H,$$

then

$$\overline{P} \ll H. \tag{4}$$

Using Dirichlet's Box we construct an integer polynomial Q such that

$$|b_n| \ll H,\ \ldots,\ |b_2| \ll H, \quad |b_1| \ll H^{1+\epsilon}, \quad |b_0| \ll H^{1}$$

$$|Q(\xi)| \ll H^{-n-\epsilon}, \tag{5}$$

If $\max(|b_1|,\ |b_0|) \gg H$, **then**

$$|\xi - \beta| \ll H(\beta)^{-\frac{n+1+2\epsilon}{1+\epsilon}}.$$

If $\max(|b_1|,\ |b_0|) \ll H$, **then**

$$\overline{Q} \ll H. \tag{6}$$

Then we can apply Wirsing's Lemma:

$$|\xi - \gamma| \ll \max \begin{cases} |P(\xi)|^{\frac{1}{2}}|Q(\xi)|\overline{P}^{n-\frac{3}{2}}, \\ \\ |P(\xi)||Q(\xi)|^{\frac{1}{2}}\overline{P}^{n-\frac{3}{2}}, \end{cases}$$

Substituting (4), (5), (6), and $|P(\xi)| \ll H^{-n-\epsilon}$, **we get:**

$$|\xi - \gamma| \ll H^{-\frac{n}{2}-\frac{3}{2}-\frac{3}{2}\epsilon} \ll H(\gamma)^{-\frac{n}{2}-\frac{3}{2}-\frac{3}{2}\epsilon}.$$

Let us compare estimates in the Case A and Case B:

$$\text{Case A:}\quad |\xi - \alpha| \ll H(\alpha)^{-\frac{n+1+2\epsilon}{1+\epsilon}}$$

$$\text{Case B:}\quad |\xi - \alpha| \ll H(\alpha)^{-\frac{n}{2}-\frac{3}{2}-\frac{3}{2}\epsilon}$$

If we take $\epsilon = 0$, then

$$\text{Case A:}\quad |\xi - \alpha| \ll H(\alpha)^{-n-1}$$

$$\text{Case B:}\quad |\xi - \alpha| \ll H(\alpha)^{-\frac{n}{2}-\frac{3}{2}}$$

On the other hand, if we take $\epsilon = 2$, then

$$\text{Case A:}\quad |\xi - \alpha| \ll H(\alpha)^{-\frac{n+5}{3}}$$

$$\text{Case B:}\quad |\xi - \alpha| \ll H(\alpha)^{-\frac{n}{2}-4.5}$$

Finally, if we choose an optimal value of ϵ, namely

$$\epsilon = 1 - \frac{6}{n},$$

we obtain

$$|\xi - \alpha| \ll H(\alpha)^{-n/2+\lambda_n}, \qquad \lim_{n\to\infty} \lambda_n = 3,$$

in both cases.

VI. “Improvement”

Let us consider an integer polynomial P such that

$$|a_n| \ll H,\ \ldots,\ |a_3| \ll H,$$
$$|a_2| \ll H^{1+\epsilon}, \quad |a_1| \ll H^{1+\epsilon}, \quad |a_0| \ll H^{1+\epsilon},$$
$$|P(\xi)| \ll H^{-n-2\epsilon}.$$

We have

$$\text{Case A:} \quad |\xi - \alpha| \ll H(\alpha)^{-\frac{n+1+3\epsilon}{1+\epsilon}}$$

$$\text{Case B:} \quad |\xi - \alpha| \ll H(\alpha)^{-\frac{n}{2}-\frac{3}{2}-3\epsilon}.$$

Put

$$\epsilon = 1 - \frac{10}{n},$$

then

$$|\xi - \alpha| \ll H(\alpha)^{-n/2+\lambda_n}, \quad \lim_{n\to\infty} \lambda_n = 4.5,$$

in both cases.

However, the Case A does not work. In fact,

$$\max(|a_2|,\ |a_1|,\ |a_0|) \gg H \quad \not\Rightarrow \quad |P'(\xi)| \quad \text{is “big”}.$$

VII. Method of "Polynomial Staircase"

In 1996 a new approach to this problem was introduced:

Step 1. Let $R^{(k)}$ be a polynomial with k "big" coefficients. We construct the following n polynomials

$$Q^{(3)}, \dots, Q^{(n+1)},\ P^{(n+1)},$$

which are small at ξ.

Step 2. We prove that they are linearly independent.

Step 3. Using a linear combination of these polynomials, we construct the polynomial

$$L^{(2)} = d_1 Q^{(3)} + \ldots + d_{n-1} Q^{(n+1)} + d_n P^{(n+1)}$$

with two "big" coefficients. The Case A does work for L. Moreover, it is possible to show that an influence of the numbers $d_1, \dots, d_n$ is very weak, so

$$|L(\xi)| \ll H^{-n-2\epsilon}.$$

THEOREM 8. **For any real number $\boldsymbol{\xi} \notin \boldsymbol{A_n}$ there exist infinitely many algebraic numbers $\boldsymbol{\alpha} \in \boldsymbol{A_n}$ with**

$$|\xi - \alpha| \ll H(\alpha)^{-\frac{n}{2}-\lambda_n}, \qquad \lim_{n\to\infty} \lambda_n = 4.$$

The following table contains the values of

$$\frac{n}{2} + \lambda_n$$

corresponding to Wirsing's Theorem (1961), the Theorem of Bernik-Tsishchanka (1993), Theorem 8 (2001), and the Conjecture:

n	1961	1993	2001	Conj.
3	3.28	3.5	3.73	4
4	3.82	4.12	4.45	5
5	4.35	4.71	5.14	6
10	6.92	7.47	8.06	11
50	26.98	27.84	28.70	51
100	51.99	52.92	53.84	101

VIII. Complex case

THEOREM 9 **(Wirsing, 1961). For any complex number $\xi \notin A_n$ there exist infinitely many algebraic numbers $\alpha \in A_n$ with**

$$|\xi - \alpha| \ll H(\alpha)^{-A},$$

where

$$A = \frac{n}{4} + 1.$$

Method: **"Resultant"**

In 2000 this result was slightly improved:

$$\boldsymbol{A = \frac{n}{4} + \lambda_n, \quad \text{where} \quad \lim_{n\to\infty} \lambda_n = \frac{3}{2}.}$$

Method: **"Big Derivative".**

Method "Polynomial Staircase": ? ? ?

IX. P-adic case

THEOREM 10 (Morrison, 1978). **Let $\xi \in \mathbb{Q}_p$. If $\xi \notin A_n$, then there are infinitely many algebraic numbers $\alpha \in A_n$ with**

$$|\xi - \alpha| \ll H(\alpha)^{-A},$$

where

$$A = \begin{cases} 1+\sqrt{3} & \text{when} \quad n = 2, \\ \dfrac{n}{2} + \dfrac{3}{2} & \text{when} \quad n > 2. \end{cases}$$

THEOREM 11 (Teulié, 2002). If $\xi \notin A_2$, then there are infinitely many algebraic numbers $\alpha \in A_2$ with

$$|\xi - \alpha| \ll H(\alpha)^{-3}.$$

The second part of Morrison's theorem was also improved:

$$A = \frac{n}{2} + \lambda_n, \quad \text{where} \quad \lim_{n \to \infty} \lambda_n = 3.$$

Method: "Big Derivative".

Method "Polynomial Staircase": ? ? ?

X. Two Counter-Examples

1. Simultaneous case.

CONJECTURE. **For any two real numbers $\xi_1, \xi_2 \notin A_n$ there exist infinitely many algebraic numbers α_1, α_2 such that**

$$\begin{cases} |\xi_1 - \alpha_1| \ll \lceil P \rceil^{-(n+1)/2}, \\ |\xi_2 - \alpha_2| \ll \lceil P \rceil^{-(n+1)/2}, \end{cases}$$

where $P(x) \in Z[x]$, $P(\alpha_1) = P(\alpha_2) = 0$, $\deg P \le n$. The implicit constant in $\ll$ should depend on ξ_1, ξ_2, and n.

Counter-example (Roy-Waldschmidt, 2001). **For any sufficiently large n there exist real numbers ξ_1 and ξ_2 such that**

$$\max\{|\xi_1 - \alpha_1|, |\xi_2 - \alpha_2|\} > \lceil P \rceil^{-3\sqrt{n}}.$$

Theorem 12. **For any real numbers $\xi_1,\ \xi_2 \notin A_n$ at least one of the following assertions is true:**

(i) There exist infinitely many algebraic numbers $\alpha_1,\ \alpha_2$ of degree $\leq n$ such that

$$\begin{cases} |\xi_1 - \alpha_1| \ll \lceil P \rceil^{-\frac{n}{8}-\frac{3}{8}}, \\ |\xi_2 - \alpha_2| \ll \lceil P \rceil^{-\frac{n}{8}-\frac{3}{8}}. \end{cases}$$

(ii) For some $\xi \in \{\xi_1,\ \xi_2\}$ there exist infinitely many algebraic numbers α of degree $2 \leq k \leq \frac{n+2}{4}$ such that

$$|\xi - \alpha| \ll H(\alpha)^{-\frac{n+2}{4}-1}.$$

2. Approximation by algebraic integers.

THEOREM 13. **(Davenport - Schmidt, 1968) Let $n \geq 3$. Let ξ be real, but not algebraic of degree ≤ 2. Then there are infinitely many algebraic integers α of degree ≤ 3 which satisfy**

$$0 < |\xi - \alpha| \ll H(\alpha)^{-\eta_3},$$

where

$$\eta_3 = \frac{1}{2}(3 + \sqrt{5}) = 2.618...$$

CONJECTURE. **Let ξ be real, but is not algebraic of degree $\leq n$. Suppose $\epsilon > 0$. Then there are infinitely many real algebraic integers α of degree $\leq n$ with**

$$|\xi - \alpha| \ll H(\alpha)^{-n+\epsilon}.$$

THEOREM 14 **(Roy, 2001). There exist real numbers ξ such that for any algebraic integer α of degree ≤ 3, we have**

$$|\xi - \alpha| \gg H(\alpha)^{-\eta_3}.$$

XI. Most Recent Result

THEOREM 15. **For any real number $\xi \notin A_3$ there exist infinitely many algebraic numbers $\alpha \in A_3$ such that**

$$|\xi - \alpha| \ll H(\alpha)^{-A},$$

where $A = 3.7475..$ is the largest root of the equation

$$2x^3 - 11x^2 + 11x + 8 = 0.$$