

On approximation of real numbers by algebraic numbers of bounded degree

BY

K. I. TSISHCHANKA

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THEOREM 1': For any real irrational number ξ there exist infinitely many polynomials $P(x) = ax + b$ with integer coefficients such that

$$|P(\xi)| < c(\xi) \overline{P}^{-1}, \quad \overline{P} = \max\{|a|, |b|\}.$$

THEOREM 2: For any real number ξ which is not rational or quadratic irrational, there exist infinitely many polynomials $P(x) = ax^2 + bx + c$ with integer coefficients such that

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DEFINITION: If α is a root of the polynomial equation

$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 = 0,$$

where a_i 's are integers and α satisfies no similar equation of degree $< n$, then α is an algebraic number of degree n .

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EXAMPLE:

number	polynomial	degree
$\frac{p}{q}$	$qx - p$	1
$\frac{1 + \sqrt{3}}{2}$	$2x^2 - 2x - 1$	2
$\sqrt{2} + \sqrt{3}$	$x^4 - 10x + 1$	4

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EXAMPLE: A_1 is the set of all rational numbers. A_2 is the set of all rational numbers and quadratic irrationals, etc.

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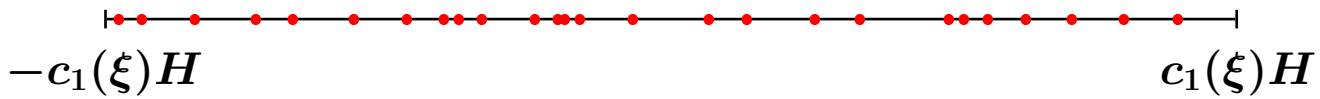
$$\begin{array}{c} |-----| \\ -c_1(\xi)H \qquad \qquad \qquad c_1(\xi)H \end{array}$$

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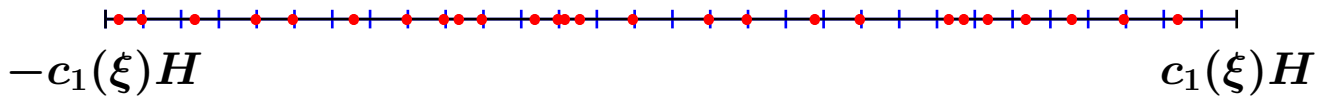


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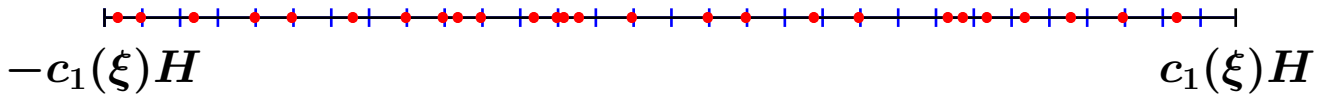
$$\text{Number of subintervals} = (2H + 1)^{n+1} - 1$$

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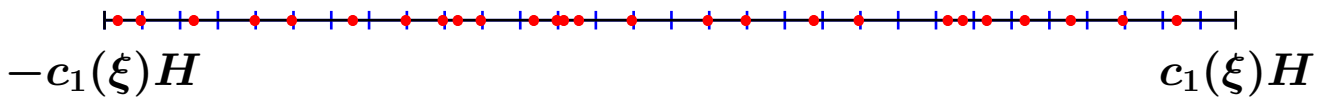
$$\text{Length} = \frac{2c_1(\xi)H}{(2H + 1)^{n+1} - 1} < \frac{2c_1(\xi)H}{H^{n+1}} = c_2(\xi)H^{-n}$$

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By the Pigeonhole Principle there exist P_1 and P_2 with

$$\underbrace{|P_1(\xi) - P_2(\xi)|}_{P(\xi)} \leq c_2(\xi)H^{-n} \leq c_2(\xi)\overline{P}^{-n} \blacksquare$$

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The implicit constant in \ll depends on ξ and n only.

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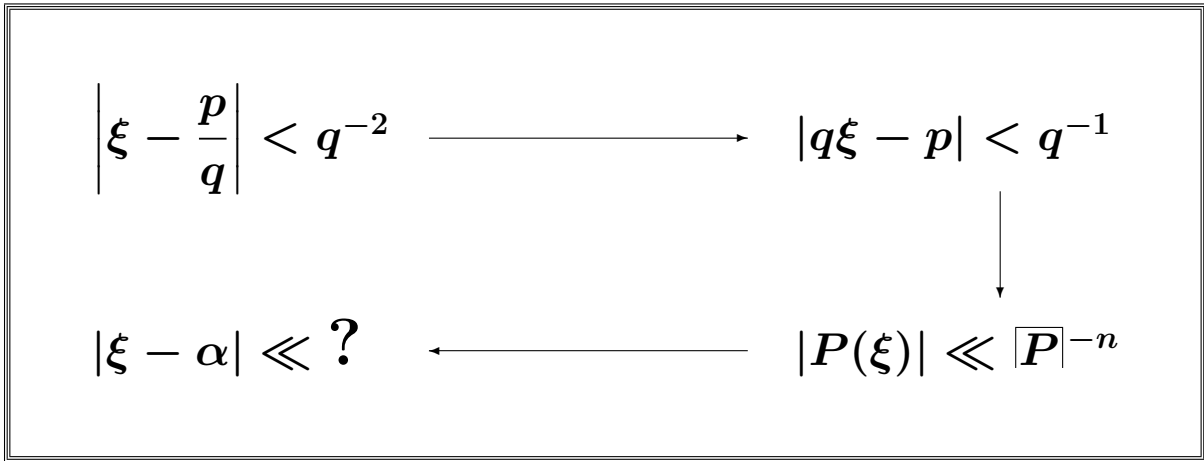
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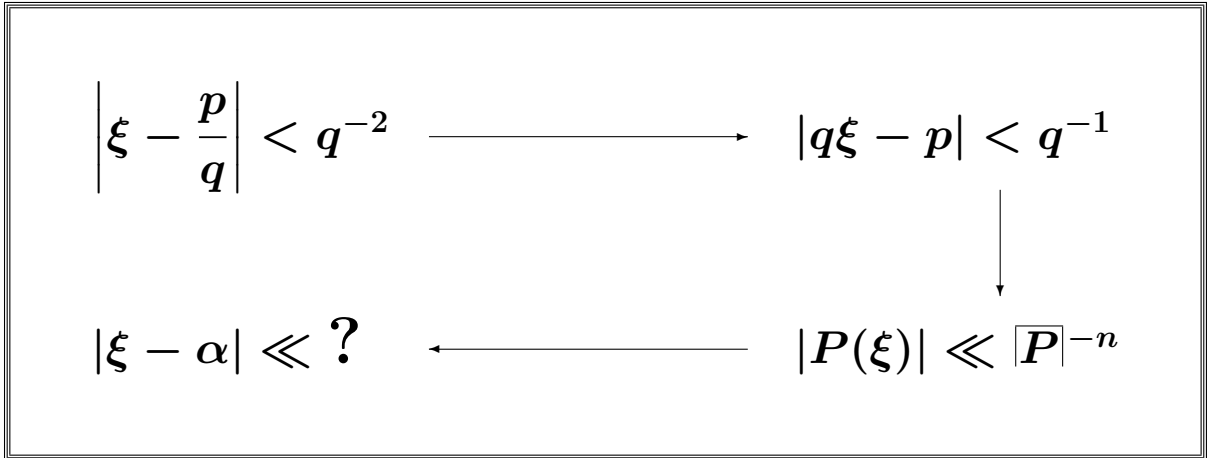


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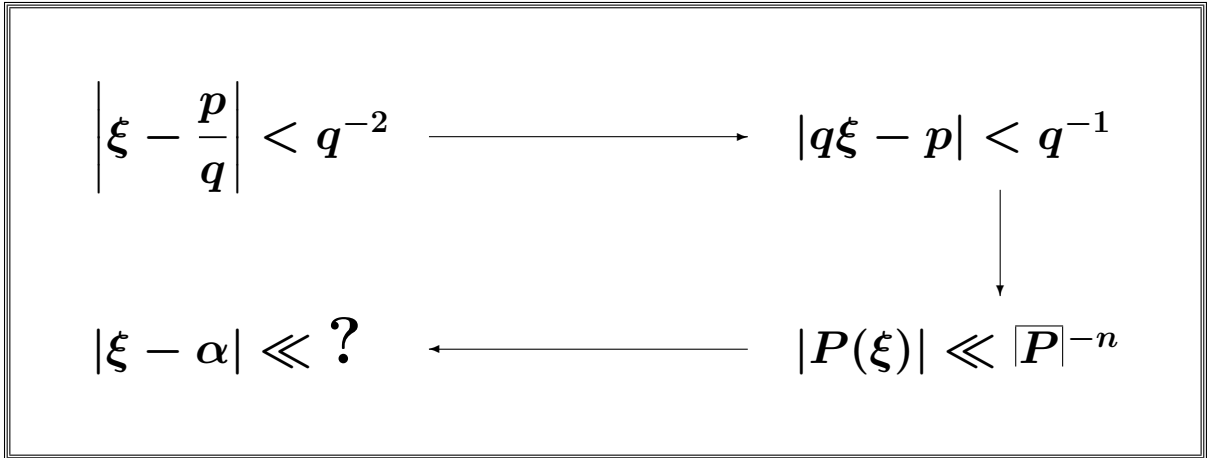


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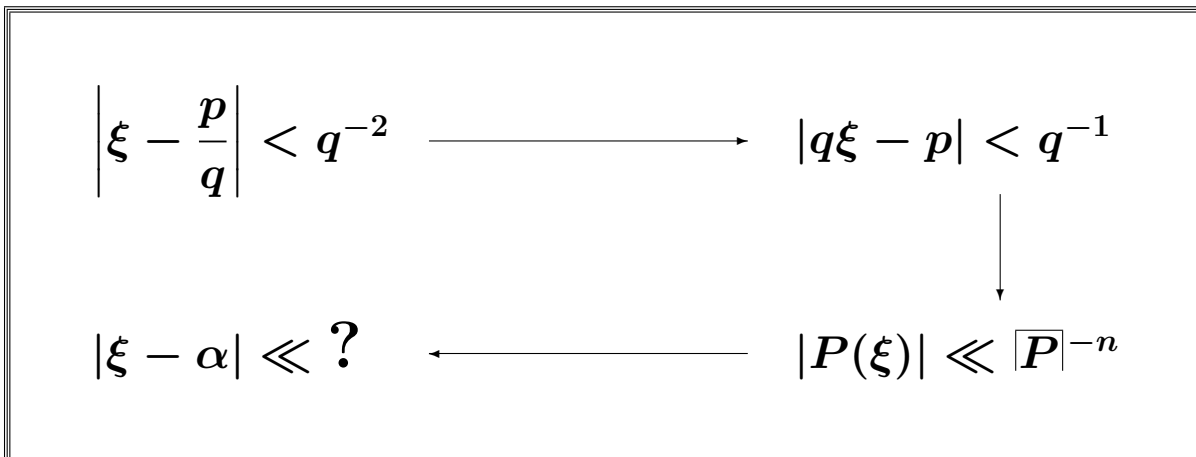
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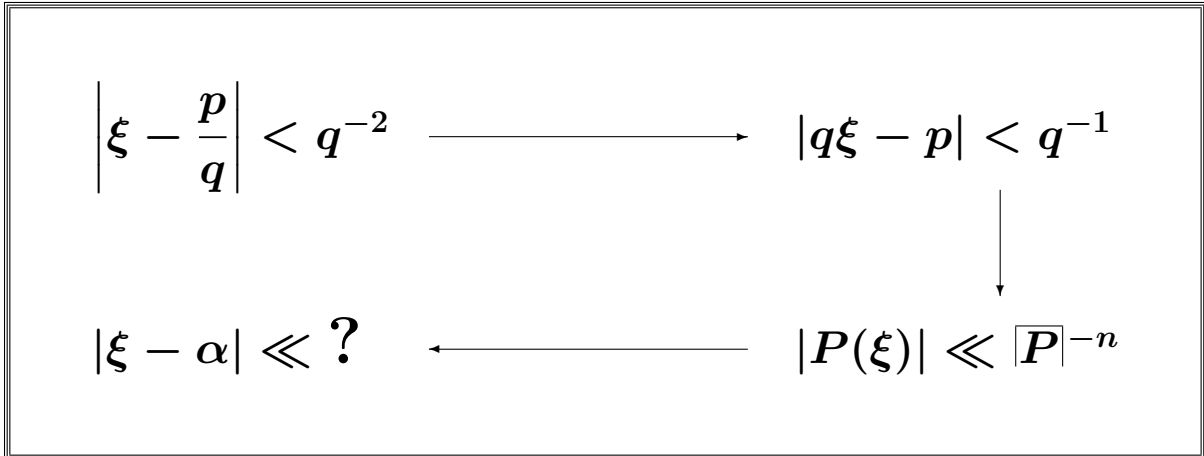


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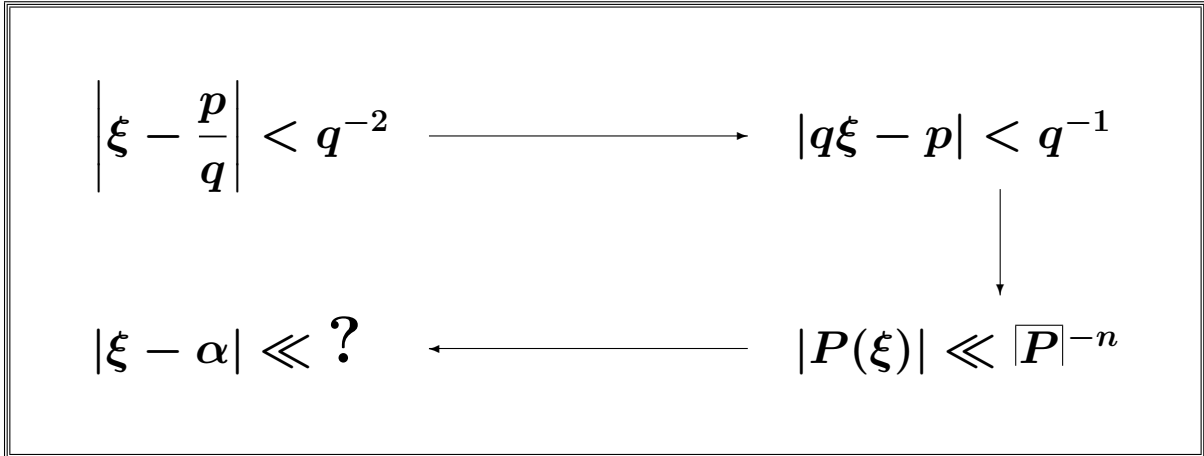
$$|x^2 - 1000x + 1000| \ll 1000^{-2} \quad \text{at} \quad \xi = 1.001002003\dots$$



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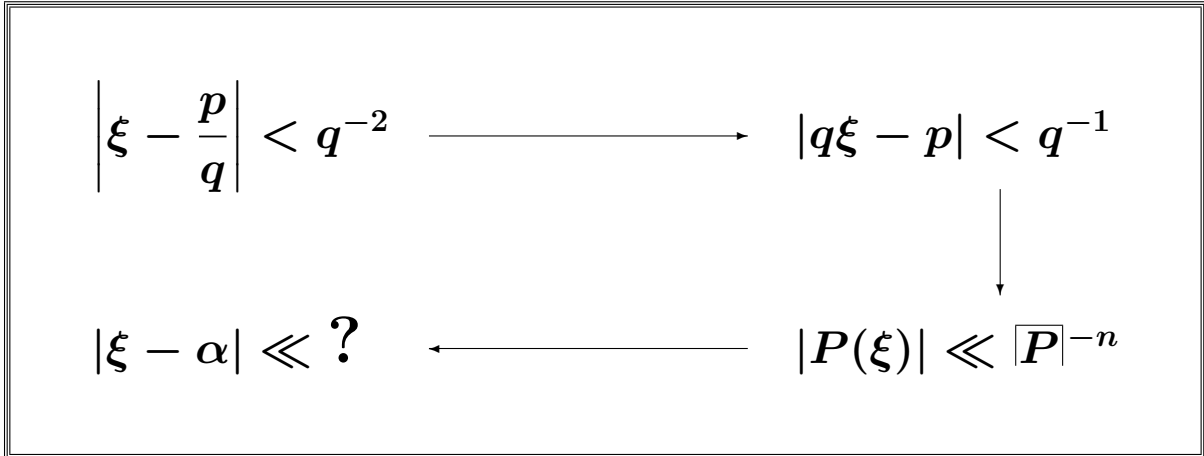
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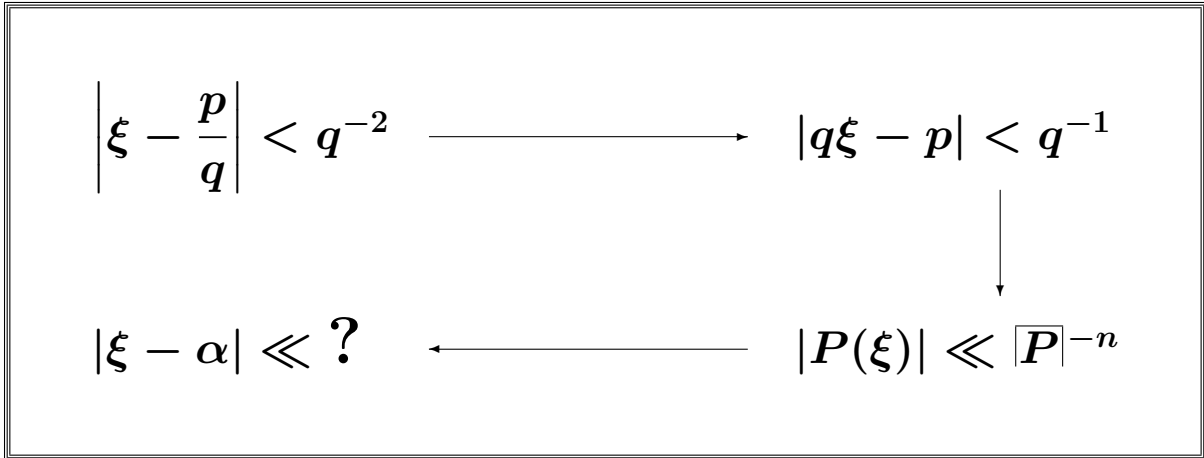


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$$n > 2 \Rightarrow |\xi - \alpha| \ll H(\alpha)^{-\frac{n}{2} - \frac{3}{2}} \text{ (Wirsing, 1961)}$$

THEOREM 5 (Wirsing, 1961): For any real number $\xi \notin A_n$ there exist infinitely many algebraic numbers $\alpha \in A_n$ with

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$$|\xi - \alpha| \ll \frac{|P(\xi)|}{|P'(\xi)|} \tag{1}$$

where α is the root of P closest to ξ .

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where α is the root of P closest to ξ .

PROOF: We get

$$\frac{|P'(\xi)|}{|P(\xi)|} = \left| \sum_{i=1}^n \frac{1}{\xi - \alpha_i} \right|$$

Let $n = 2$, then

$$\frac{|P'(x)|}{|P(x)|} = \frac{|[a_2(x - \alpha_1)(x - \alpha_2)]'|}{|a_2(x - \alpha_1)(x - \alpha_2)|} = \frac{|(x - \alpha_1) + (x - \alpha_2)|}{|(x - \alpha_1)(x - \alpha_2)|}$$

THEOREM 5 (Wirsing, 1961): For any real number $\xi \notin A_n$ there exist infinitely many algebraic numbers $\alpha \in A_n$ with

$$|\xi - \alpha| \ll H(\alpha)^{-\frac{n}{2} - \frac{3}{2}}$$

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PROOF OF THE THEOREM: Assume to the contrary that there exists a real number $\xi \notin A_n$ such that

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$$H(\alpha)^{-\omega} \ll \frac{|P(\xi)|}{|P'(\xi)|} \quad (3)$$

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$$|P'(\xi)| \ll |P(\xi)| |P|^\omega \quad (3)$$

LEMMA 2: There are infinitely many polynomials $P, Q \in \mathbb{Z}[x]$ of degree $\leq n$, such that

$$|P(\xi)| \ll |P|^{-n}$$

$$|Q(\xi)| \ll |P|^{-n}$$

$$|Q| \ll |P|$$

and

P, Q have no
common root

LEMMA 3: Let $P, Q \in Z[x]$ be polynomials of degree d with $1 < d \leq n$. Suppose that P and Q have no common root. Then at least one of the following estimates is true:

$$1 \ll \max \{|P(\xi)|, |Q(\xi)|\}^2 \max \{\overline{P}, \overline{Q}\}^{2n-2}$$

$$1 \ll \max \{|P(\xi)||P'(\xi)||Q'(\xi)|, |Q(\xi)||P'(\xi)|^2\} \overline{P}^{n-2} \overline{Q}^{n-1}$$

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$$\ll \overline{P}^{-2}$$

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LEMMA 3: Let $P, Q \in Z[x]$ be polynomials of degree d with $1 < d \leq n$. Suppose that P and Q have no common root. Then at least one of the following estimates is true:

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$$\ll \overline{P}^{2\omega-n-3} \Rightarrow 2\omega - n - 3 > 0$$

LEMMA 3: Let $P, Q \in Z[x]$ be polynomials of degree d with $1 < d \leq n$. Suppose that P and Q have no common root. Then at least one of the following estimates is true:

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$$\ll \overline{P}^{2\omega-n-3} \Rightarrow 2\omega - n - 3 > 0 \Rightarrow \omega > \frac{n+3}{2}$$

LEMMA 3: Let $P, Q \in Z[x]$ be polynomials of degree d with $1 < d \leq n$. Suppose that P and Q have no common root. Then at least one of the following estimates is true:

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PROOF: Let

$$P(x) = a_\ell x^\ell + \dots + a_1 x + a_0 = a_\ell (x - \alpha_1) \dots (x - \alpha_\ell)$$

$$Q(x) = b_m x^m + \dots + b_1 x + b_0 = b_m (x - \beta_1) \dots (x - \beta_m)$$

Consider

$$a_\ell^m b_m^\ell \prod_{\substack{1 \leq i \leq \ell \\ 1 \leq j \leq m}} (\alpha_i - \beta_j)$$

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Consider

$$R(P, Q) = a_\ell^m b_m^\ell \prod_{\substack{1 \leq i \leq \ell \\ 1 \leq j \leq m}} (\alpha_i - \beta_j)$$

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We have

$$R(P, Q) = a_\ell^m b_m^\ell \prod_{\substack{1 \leq i \leq \ell \\ 1 \leq j \leq m}} (\alpha_i - \beta_j) \neq 0$$

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$$|R(P, Q)| = |a_\ell^m b_m^\ell \prod_{\substack{1 \leq i \leq \ell \\ 1 \leq j \leq m}} (\alpha_i - \beta_j)| > 0$$

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We have

$$|R(P, Q)| = |a_\ell^m b_m^\ell \prod_{\substack{1 \leq i \leq \ell \\ 1 \leq j \leq m}} (\alpha_i - \beta_j)| > 0$$

$$R(P, Q) = \left| \begin{array}{cccc} a_\ell & \dots & a_1 & a_0 \\ \dots & & \dots & \dots \\ & a_\ell & \dots & a_1 & a_0 \\ b_m & \dots & b_1 & b_0 & \\ \dots & & \dots & \dots & \\ & b_m & \dots & b_1 & b_0 \end{array} \right| \begin{array}{l} \left. \vphantom{\begin{array}{c} a_\ell \\ \dots \\ a_\ell \\ b_m \\ \dots \\ b_m \end{array}} \right\} m \\ \left. \vphantom{\begin{array}{c} a_1 \\ \dots \\ a_1 \\ b_1 \\ \dots \\ b_1 \end{array}} \right\} \ell \end{array}$$

PROOF: Let

$$P(x) = a_\ell x^\ell + \dots + a_1 x + a_0 = a_\ell (x - \alpha_1) \dots (x - \alpha_\ell)$$

$$Q(x) = b_m x^m + \dots + b_1 x + b_0 = b_m (x - \beta_1) \dots (x - \beta_m)$$

We have

$$|R(P, Q)| = |a_\ell^m b_m^\ell \prod_{\substack{1 \leq i \leq \ell \\ 1 \leq j \leq m}} (\alpha_i - \beta_j)| \geq 1$$

PROOF: Let

$$P(x) = a_\ell x^\ell + \dots + a_1 x + a_0 = a_\ell (x - \alpha_1) \dots (x - \alpha_\ell)$$
$$Q(x) = b_m x^m + \dots + b_1 x + b_0 = b_m (x - \beta_1) \dots (x - \beta_m)$$

We have

$$|R(P, Q)| = |a_\ell^m b_m^\ell \prod_{\substack{1 \leq i \leq \ell \\ 1 \leq j \leq m}} (\alpha_i - \beta_j)| \geq 1$$

$$1 \ll \max \{|P(\xi)|, |Q(\xi)|\}^2 \max \{\overline{P}, \overline{Q}\}^{2n-2}$$

$$1 \ll \max \{|P(\xi)| |P'(\xi)| |Q'(\xi)|, |Q(\xi)| |P'(\xi)|^2\} \overline{P}^{n-2} \overline{Q}^{n-1}$$

$$1 \ll \max \{|Q(\xi)| |P'(\xi)| |Q'(\xi)|, |P(\xi)| |Q'(\xi)|^2\} \overline{P}^{n-1} \overline{Q}^{n-2}$$

PROOF: Let

$$P(x) = a_\ell x^\ell + \dots + a_1 x + a_0 = a_\ell (x - \alpha_1) \dots (x - \alpha_\ell)$$

$$Q(x) = b_m x^m + \dots + b_1 x + b_0 = b_m (x - \beta_1) \dots (x - \beta_m)$$

We have

$$|R(P, Q)| = |a_\ell^m b_m^\ell \prod_{\substack{1 \leq i \leq \ell \\ 1 \leq j \leq m}} (\alpha_i - \beta_j)| \geq 1$$

$$\begin{vmatrix} a_\ell & \dots & a_1 & a_0 \\ \dots & & \dots & \dots \\ & a_\ell & \dots & a_1 & a_0 \\ b_m & \dots & b_1 & b_0 \\ \dots & & \dots & \dots \\ & b_m & \dots & b_1 & b_0 \end{vmatrix} \equiv \begin{vmatrix} \frac{P^{(\ell)}(0)}{\ell!} & \dots & P'(0) & P(0) \\ & \dots & \dots & \dots \\ & & \frac{P^{(\ell)}(0)}{\ell!} & \dots & P'(0) & P(0) \\ \frac{Q^{(m)}(0)}{m!} & \dots & Q'(0) & Q(0) \\ & \dots & \dots & \dots \\ & & \frac{Q^{(m)}(0)}{m!} & \dots & Q'(0) & Q(0) \end{vmatrix}$$

PROOF: Let

$$P(x) = a_\ell x^\ell + \dots + a_1 x + a_0 = a_\ell (x - \alpha_1) \dots (x - \alpha_\ell)$$

$$Q(x) = b_m x^m + \dots + b_1 x + b_0 = b_m (x - \beta_1) \dots (x - \beta_m)$$

We have

$$|R(P, Q)| = |a_\ell^m b_m^\ell \prod_{\substack{1 \leq i \leq \ell \\ 1 \leq j \leq m}} (\alpha_i - \beta_j)| \geq 1$$

$$\begin{vmatrix} a_\ell & \dots & a_1 & a_0 \\ \dots & & \dots & \dots \\ & a_\ell & \dots & a_1 & a_0 \\ b_m & \dots & b_1 & b_0 \\ \dots & & \dots & \dots \\ & b_m & \dots & b_1 & b_0 \end{vmatrix} \equiv \begin{vmatrix} \frac{P^{(\ell)}(\xi)}{\ell!} & \dots & P'(\xi) & P(\xi) \\ & \dots & \dots & \dots \\ & & \frac{P^{(\ell)}(\xi)}{\ell!} & \dots & P'(\xi) & P(\xi) \\ \frac{Q^{(m)}(\xi)}{m!} & \dots & Q'(\xi) & Q(\xi) \\ & \dots & \dots & \dots \\ & & \frac{Q^{(m)}(\xi)}{m!} & \dots & Q'(\xi) & Q(\xi) \end{vmatrix}$$

PROOF: Let

$$P(x) = a_\ell x^\ell + \dots + a_1 x + a_0 = a_\ell (x - \alpha_1) \dots (x - \alpha_\ell)$$

$$Q(x) = b_m x^m + \dots + b_1 x + b_0 = b_m (x - \beta_1) \dots (x - \beta_m)$$

We have

$$|R(P, Q)| = |a_\ell^m b_m^\ell \prod_{\substack{1 \leq i \leq \ell \\ 1 \leq j \leq m}} (\alpha_i - \beta_j)| \geq 1$$

$$\left\| \begin{array}{cccc} a_\ell & \dots & a_1 & a_0 \\ \dots & & \dots & \dots \\ & a_\ell & \dots & a_1 & a_0 \\ b_m & \dots & b_1 & b_0 \\ \dots & & \dots & \dots \\ & b_m & \dots & b_1 & b_0 \end{array} \right\| \equiv \left\| \begin{array}{cccc} \frac{P^{(\ell)}(\xi)}{\ell!} & \dots & P'(\xi) & P(\xi) \\ & \dots & \dots & \dots \\ & & \frac{P^{(\ell)}(\xi)}{\ell!} & \dots & P'(\xi) & P(\xi) \\ \frac{Q^{(m)}(\xi)}{m!} & \dots & Q'(\xi) & Q(\xi) \\ & \dots & \dots & \dots \\ & & \frac{Q^{(m)}(\xi)}{m!} & \dots & Q'(\xi) & Q(\xi) \end{array} \right\|$$

$$\ll \max \{|P(\xi)|, |Q(\xi)|\}^2 \max \{\overline{P}, \overline{Q}\}^{2n-2}$$

PROOF: Let

$$P(x) = a_\ell x^\ell + \dots + a_1 x + a_0 = a_\ell (x - \alpha_1) \dots (x - \alpha_\ell)$$

$$Q(x) = b_m x^m + \dots + b_1 x + b_0 = b_m (x - \beta_1) \dots (x - \beta_m)$$

We have

$$|R(P, Q)| = |a_\ell^m b_m^\ell \prod_{\substack{1 \leq i \leq \ell \\ 1 \leq j \leq m}} (\alpha_i - \beta_j)| \geq 1$$

$$\left\| \begin{array}{cccc} a_\ell & \dots & a_1 & a_0 \\ \dots & & \dots & \dots \\ & a_\ell & \dots & a_1 & a_0 \\ b_m & \dots & b_1 & b_0 \\ \dots & & \dots & \dots \\ & b_m & \dots & b_1 & b_0 \end{array} \right\| \equiv \left\| \begin{array}{cccc} \frac{P^{(\ell)}(\xi)}{\ell!} & \dots & P'(\xi) & P(\xi) \\ & \dots & \dots & \dots \\ & & \frac{P^{(\ell)}(\xi)}{\ell!} & \dots & P'(\xi) & P(\xi) \\ \frac{Q^{(m)}(\xi)}{m!} & \dots & Q'(\xi) & Q(\xi) \\ & \dots & \dots & \dots \\ & & \frac{Q^{(m)}(\xi)}{m!} & \dots & Q'(\xi) & Q(\xi) \end{array} \right\|$$

$$\ll \max \{ |P(\xi)| |P'(\xi)| |Q'(\xi)|, |Q(\xi)| |P'(\xi)|^2 \} |P|^{n-2} |Q|^{n-1}$$

PROOF: Let

$$P(x) = a_\ell x^\ell + \dots + a_1 x + a_0 = a_\ell (x - \alpha_1) \dots (x - \alpha_\ell)$$

$$Q(x) = b_m x^m + \dots + b_1 x + b_0 = b_m (x - \beta_1) \dots (x - \beta_m)$$

We have

$$|R(P, Q)| = |a_\ell^m b_m^\ell \prod_{\substack{1 \leq i \leq \ell \\ 1 \leq j \leq m}} (\alpha_i - \beta_j)| \geq 1$$

$$\left\| \begin{array}{cccc} a_\ell & \dots & a_1 & a_0 \\ \dots & & \dots & \dots \\ & a_\ell & \dots & a_1 & a_0 \\ b_m & \dots & b_1 & b_0 \\ \dots & & \dots & \dots \\ & b_m & \dots & b_1 & b_0 \end{array} \right\| \equiv \left\| \begin{array}{cccc} \frac{P^{(\ell)}(\xi)}{\ell!} & \dots & P'(\xi) & P(\xi) \\ & \dots & \dots & \dots \\ & & \frac{P^{(\ell)}(\xi)}{\ell!} & \dots & P'(\xi) & P(\xi) \\ \frac{Q^{(m)}(\xi)}{m!} & \dots & Q'(\xi) & Q(\xi) \\ & \dots & \dots & \dots \\ & & \frac{Q^{(m)}(\xi)}{m!} & \dots & Q'(\xi) & Q(\xi) \end{array} \right\|$$

$$\ll \max \{ |Q(\xi)| |P'(\xi)| |Q'(\xi)|, |P(\xi)| |Q'(\xi)|^2 \} |P|^{n-1} |Q|^{n-2}$$

$\frac{P^{(5)}(\xi)}{5!}$	$\frac{P^{(4)}(\xi)}{4!}$	$\frac{P^{(3)}(\xi)}{3!}$	$\frac{P''(\xi)}{2!}$	$P'(\xi)$	$P(\xi)$	0	0	0	0
0	$\frac{P^{(5)}(\xi)}{5!}$	$\frac{P^{(4)}(\xi)}{4!}$	$\frac{P^{(3)}(\xi)}{3!}$	$\frac{P''(\xi)}{2!}$	$P'(\xi)$	$P(\xi)$	0	0	0
0	0	$\frac{P^{(5)}(\xi)}{5!}$	$\frac{P^{(4)}(\xi)}{4!}$	$\frac{P^{(3)}(\xi)}{3!}$	$\frac{P''(\xi)}{2!}$	$P'(\xi)$	$P(\xi)$	0	0
0	0	0	$\frac{P^{(5)}(\xi)}{5!}$	$\frac{P^{(4)}(\xi)}{4!}$	$\frac{P^{(3)}(\xi)}{3!}$	$\frac{P''(\xi)}{2!}$	$P'(\xi)$	$P(\xi)$	0
0	0	0	0	$\frac{P^{(5)}(\xi)}{5!}$	$\frac{P^{(4)}(\xi)}{4!}$	$\frac{P^{(3)}(\xi)}{3!}$	$\frac{P''(\xi)}{2!}$	$P'(\xi)$	$P(\xi)$
$\frac{Q^{(5)}(\xi)}{5!}$	$\frac{Q^{(4)}(\xi)}{4!}$	$\frac{Q^{(3)}(\xi)}{3!}$	$\frac{Q''(\xi)}{2!}$	$Q'(\xi)$	$Q(\xi)$	0	0	0	0
0	$\frac{Q^{(5)}(\xi)}{5!}$	$\frac{Q^{(4)}(\xi)}{4!}$	$\frac{Q^{(3)}(\xi)}{3!}$	$\frac{Q''(\xi)}{2!}$	$Q'(\xi)$	$Q(\xi)$	0	0	0
0	0	$\frac{Q^{(5)}(\xi)}{5!}$	$\frac{Q^{(4)}(\xi)}{4!}$	$\frac{Q^{(3)}(\xi)}{3!}$	$\frac{Q''(\xi)}{2!}$	$Q'(\xi)$	$Q(\xi)$	0	0
0	0	0	$\frac{Q^{(5)}(\xi)}{5!}$	$\frac{Q^{(4)}(\xi)}{4!}$	$\frac{Q^{(3)}(\xi)}{3!}$	$\frac{Q''(\xi)}{2!}$	$Q'(\xi)$	$Q(\xi)$	0
0	0	0	0	$\frac{Q^{(5)}(\xi)}{5!}$	$\frac{Q^{(4)}(\xi)}{4!}$	$\frac{Q^{(3)}(\xi)}{3!}$	$\frac{Q''(\xi)}{2!}$	$Q'(\xi)$	$Q(\xi)$

$\frac{P^{(5)}(\xi)}{5!}$	$\frac{P^{(4)}(\xi)}{4!}$	$\frac{P^{(3)}(\xi)}{3!}$	$\frac{P''(\xi)}{2!}$	$P'(\xi)$	$P(\xi)$	0	0	0	0
0	$\frac{P^{(5)}(\xi)}{5!}$	$\frac{P^{(4)}(\xi)}{4!}$	$\frac{P^{(3)}(\xi)}{3!}$	$\frac{P''(\xi)}{2!}$	$P'(\xi)$	$P(\xi)$	0	0	0
0	0	$\frac{P^{(5)}(\xi)}{5!}$	$\frac{P^{(4)}(\xi)}{4!}$	$\frac{P^{(3)}(\xi)}{3!}$	$\frac{P''(\xi)}{2!}$	$P'(\xi)$	$P(\xi)$	0	0
0	0	0	$\frac{P^{(5)}(\xi)}{5!}$	$\frac{P^{(4)}(\xi)}{4!}$	$\frac{P^{(3)}(\xi)}{3!}$	$\frac{P''(\xi)}{2!}$	$P'(\xi)$	$P(\xi)$	0
0	0	0	0	$\frac{P^{(5)}(\xi)}{5!}$	$\frac{P^{(4)}(\xi)}{4!}$	$\frac{P^{(3)}(\xi)}{3!}$	$\frac{P''(\xi)}{2!}$	$P'(\xi)$	$P(\xi)$
$\frac{Q^{(5)}(\xi)}{5!}$	$\frac{Q^{(4)}(\xi)}{4!}$	$\frac{Q^{(3)}(\xi)}{3!}$	$\frac{Q''(\xi)}{2!}$	$Q'(\xi)$	$Q(\xi)$	0	0	0	0
0	$\frac{Q^{(5)}(\xi)}{5!}$	$\frac{Q^{(4)}(\xi)}{4!}$	$\frac{Q^{(3)}(\xi)}{3!}$	$\frac{Q''(\xi)}{2!}$	$Q'(\xi)$	$Q(\xi)$	0	0	0
0	0	$\frac{Q^{(5)}(\xi)}{5!}$	$\frac{Q^{(4)}(\xi)}{4!}$	$\frac{Q^{(3)}(\xi)}{3!}$	$\frac{Q''(\xi)}{2!}$	$Q'(\xi)$	$Q(\xi)$	0	0
0	0	0	$\frac{Q^{(5)}(\xi)}{5!}$	$\frac{Q^{(4)}(\xi)}{4!}$	$\frac{Q^{(3)}(\xi)}{3!}$	$\frac{Q''(\xi)}{2!}$	$Q'(\xi)$	$Q(\xi)$	0
0	0	0	0	$\frac{Q^{(5)}(\xi)}{5!}$	$\frac{Q^{(4)}(\xi)}{4!}$	$\frac{Q^{(3)}(\xi)}{3!}$	$\frac{Q''(\xi)}{2!}$	$Q'(\xi)$	$Q(\xi)$

\overline{P}	\overline{P}	\overline{P}	\overline{P}	\overline{P}	\overline{P}	0	0	0	0
0	\overline{P}	\overline{P}	\overline{P}	\overline{P}	\overline{P}	\overline{P}	0	0	0
0	0	\overline{P}	\overline{P}	\overline{P}	\overline{P}	\overline{P}	$P(\xi)$	0	0
0	0	0	\overline{P}	\overline{P}	\overline{P}	\overline{P}	$P'(\xi)$	$P(\xi)$	0
0	0	0	0	\overline{P}	\overline{P}	\overline{P}	$\frac{P''(\xi)}{2!}$	$P'(\xi)$	$P(\xi)$
\overline{Q}	\overline{Q}	\overline{Q}	\overline{Q}	\overline{Q}	\overline{Q}	0	0	0	0
0	\overline{Q}	\overline{Q}	\overline{Q}	\overline{Q}	\overline{Q}	\overline{Q}	0	0	0
0	0	\overline{Q}	\overline{Q}	\overline{Q}	\overline{Q}	\overline{Q}	$Q(\xi)$	0	0
0	0	0	\overline{Q}	\overline{Q}	\overline{Q}	\overline{Q}	$Q'(\xi)$	$Q(\xi)$	0
0	0	0	0	\overline{Q}	\overline{Q}	\overline{Q}	$\frac{Q''(\xi)}{2!}$	$Q'(\xi)$	$Q(\xi)$

\overline{P}	\overline{P}	\overline{P}	\overline{P}	\overline{P}	\overline{P}	0	0	0	0
0	\overline{P}	\overline{P}	\overline{P}	\overline{P}	\overline{P}	\overline{P}	0	0	0
0	0	\overline{P}	\overline{P}	\overline{P}	\overline{P}	\overline{P}	$P(\xi)$	0	0
0	0	0	\overline{P}	\overline{P}	\overline{P}	\overline{P}	$P'(\xi)$	$P(\xi)$	0
0	0	0	0	\overline{P}	\overline{P}	\overline{P}	\overline{P}	$P'(\xi)$	$P(\xi)$
\overline{Q}	\overline{Q}	\overline{Q}	\overline{Q}	\overline{Q}	\overline{Q}	0	0	0	0
0	\overline{Q}	\overline{Q}	\overline{Q}	\overline{Q}	\overline{Q}	\overline{Q}	0	0	0
0	0	\overline{Q}	\overline{Q}	\overline{Q}	\overline{Q}	\overline{Q}	$Q(\xi)$	0	0
0	0	0	\overline{Q}	\overline{Q}	\overline{Q}	\overline{Q}	$Q'(\xi)$	$Q(\xi)$	0
0	0	0	0	\overline{Q}	\overline{Q}	\overline{Q}	\overline{Q}	$Q'(\xi)$	$Q(\xi)$

\overline{P}	\overline{P}	\overline{P}	\overline{P}	\overline{P}	\overline{P}	0	0	0	0
0	\overline{P}	\overline{P}	\overline{P}	\overline{P}	\overline{P}	\overline{P}	0	0	0
0	0	\overline{P}	\overline{P}	\overline{P}	\overline{P}	\overline{P}	\overline{P}	0	0
0	0	0	\overline{P}	\overline{P}	\overline{P}	\overline{P}	$P'(\xi)$	$P(\xi)$	0
0	0	0	0	\overline{P}	\overline{P}	\overline{P}	\overline{P}	$P'(\xi)$	$P(\xi)$
\overline{Q}	\overline{Q}	\overline{Q}	\overline{Q}	\overline{Q}	\overline{Q}	0	0	0	0
0	\overline{Q}	\overline{Q}	\overline{Q}	\overline{Q}	\overline{Q}	\overline{Q}	0	0	0
0	0	\overline{Q}	\overline{Q}	\overline{Q}	\overline{Q}	\overline{Q}	\overline{Q}	0	0
0	0	0	\overline{Q}	\overline{Q}	\overline{Q}	\overline{Q}	$Q'(\xi)$	$Q(\xi)$	0
0	0	0	0	\overline{Q}	\overline{Q}	\overline{Q}	\overline{Q}	$Q'(\xi)$	$Q(\xi)$

\overline{P}	\overline{P}	\overline{P}	\overline{P}	\overline{P}	\overline{P}	0	0	0	0
0	\overline{P}	\overline{P}	\overline{P}	\overline{P}	\overline{P}	\overline{P}	0	0	0
0	0	\overline{P}	\overline{P}	\overline{P}	\overline{P}	\overline{P}	\overline{P}	0	0
0	0	0	\overline{P}	\overline{P}	\overline{P}	\overline{P}	$P'(\xi)$	$P(\xi)$	0
0	0	0	0	\overline{P}	\overline{P}	\overline{P}	\overline{P}	$P'(\xi)$	$P(\xi)$
\overline{Q}	\overline{Q}	\overline{Q}	\overline{Q}	\overline{Q}	\overline{Q}	0	0	0	0
0	\overline{Q}	\overline{Q}	\overline{Q}	\overline{Q}	\overline{Q}	\overline{Q}	0	0	0
0	0	\overline{Q}	\overline{Q}	\overline{Q}	\overline{Q}	\overline{Q}	\overline{Q}	0	0
0	0	0	\overline{Q}	\overline{Q}	\overline{Q}	\overline{Q}	$Q'(\xi)$	$Q(\xi)$	0
0	0	0	0	\overline{Q}	\overline{Q}	\overline{Q}	\overline{Q}	$Q'(\xi)$	$Q(\xi)$

$$\ll \max \{ |P(\xi)|, |Q(\xi)| \}^2 \max \{ \overline{P}, \overline{Q} \}^8$$

$$\begin{array}{ccccccccccc}
\overline{P} & \overline{P} & \overline{P} & \overline{P} & \overline{P} & \overline{P} & 0 & 0 & 0 & 0 \\
0 & \overline{P} & \overline{P} & \overline{P} & \overline{P} & \overline{P} & \overline{P} & 0 & 0 & 0 \\
0 & 0 & \overline{P} & \overline{P} & \overline{P} & \overline{P} & \overline{P} & \overline{P} & 0 & 0 \\
0 & 0 & 0 & \overline{P} & \overline{P} & \overline{P} & \overline{P} & P'(\xi) & P(\xi) & 0 \\
0 & 0 & 0 & 0 & \overline{P} & \overline{P} & \overline{P} & \overline{P} & P'(\xi) & P(\xi) \\
\overline{Q} & \overline{Q} & \overline{Q} & \overline{Q} & \overline{Q} & \overline{Q} & 0 & 0 & 0 & 0 \\
0 & \overline{Q} & \overline{Q} & \overline{Q} & \overline{Q} & \overline{Q} & \overline{Q} & 0 & 0 & 0 \\
0 & 0 & \overline{Q} & \overline{Q} & \overline{Q} & \overline{Q} & \overline{Q} & \overline{Q} & 0 & 0 \\
0 & 0 & 0 & \overline{Q} & \overline{Q} & \overline{Q} & \overline{Q} & Q'(\xi) & Q(\xi) & 0 \\
0 & 0 & 0 & 0 & \overline{Q} & \overline{Q} & \overline{Q} & \overline{Q} & Q'(\xi) & Q(\xi)
\end{array}$$

$$\ll \max \{ |P(\xi)| |P'(\xi)| |Q'(\xi)|, |Q(\xi)| |P'(\xi)|^2 \} \overline{P}^3 \overline{Q}^4$$

$$\begin{array}{ccccccccccc}
\overline{P} & \overline{P} & \overline{P} & \overline{P} & \overline{P} & \overline{P} & 0 & 0 & 0 & 0 \\
0 & \overline{P} & \overline{P} & \overline{P} & \overline{P} & \overline{P} & \overline{P} & 0 & 0 & 0 \\
0 & 0 & \overline{P} & \overline{P} & \overline{P} & \overline{P} & \overline{P} & \overline{P} & 0 & 0 \\
0 & 0 & 0 & \overline{P} & \overline{P} & \overline{P} & \overline{P} & P'(\xi) & P(\xi) & 0 \\
0 & 0 & 0 & 0 & \overline{P} & \overline{P} & \overline{P} & \overline{P} & P'(\xi) & P(\xi) \\
\overline{Q} & \overline{Q} & \overline{Q} & \overline{Q} & \overline{Q} & \overline{Q} & 0 & 0 & 0 & 0 \\
0 & \overline{Q} & \overline{Q} & \overline{Q} & \overline{Q} & \overline{Q} & \overline{Q} & 0 & 0 & 0 \\
0 & 0 & \overline{Q} & \overline{Q} & \overline{Q} & \overline{Q} & \overline{Q} & \overline{Q} & 0 & 0 \\
0 & 0 & 0 & \overline{Q} & \overline{Q} & \overline{Q} & \overline{Q} & Q'(\xi) & Q(\xi) & 0 \\
0 & 0 & 0 & 0 & \overline{Q} & \overline{Q} & \overline{Q} & \overline{Q} & Q'(\xi) & Q(\xi)
\end{array}$$

$$\ll \max \{ |Q(\xi)| |P'(\xi)| |Q'(\xi)|, |P(\xi)| |Q'(\xi)|^2 \} \overline{P}^4 \overline{Q}^3$$

PROOF: Let

$$P(x) = a_\ell x^\ell + \dots + a_1 x + a_0 = a_\ell (x - \alpha_1) \dots (x - \alpha_\ell)$$

$$Q(x) = b_m x^m + \dots + b_1 x + b_0 = b_m (x - \beta_1) \dots (x - \beta_m)$$

We have

$$|R(P, Q)| = |a_\ell^m b_m^\ell \prod_{\substack{1 \leq i \leq \ell \\ 1 \leq j \leq m}} (\alpha_i - \beta_j)| \geq 1$$

PROOF: Let

$$P(x) = a_\ell x^\ell + \dots + a_1 x + a_0 = a_\ell (x - \alpha_1) \dots (x - \alpha_\ell)$$

$$Q(x) = b_m x^m + \dots + b_1 x + b_0 = b_m (x - \beta_1) \dots (x - \beta_m)$$

We have

$$|R(P, Q)| = |a_\ell^m b_m^\ell \prod_{\substack{1 \leq i \leq \ell \\ 1 \leq j \leq m}} (\alpha_i - \beta_j)| \geq 1$$

$$|R(P, Q)| \ll \begin{cases} \max \{|P(\xi)|, |Q(\xi)|\}^2 \max \{\overline{P}, \overline{Q}\}^{2n-2} \\ \max \{|P(\xi)| |P'(\xi)| |Q'(\xi)|, |Q(\xi)| |P'(\xi)|^2\} \overline{P}^{n-2} \overline{Q}^{n-1} \\ \max \{|Q(\xi)| |P'(\xi)| |Q'(\xi)|, |P(\xi)| |Q'(\xi)|^2\} \overline{P}^{n-1} \overline{Q}^{n-2} \end{cases}$$

LEMMA 3: Let $P, Q \in Z[x]$ be polynomials of degree d with $1 < d \leq n$. Suppose that P and Q have no common root. Then at least one of the following estimates is true:

$$1 \ll \max \{ |P(\xi)|, |Q(\xi)| \}^2 \max \{ \overline{P}, \overline{Q} \}^{2n-2}$$

$$1 \ll \max \{ |P(\xi)| |P'(\xi)| |Q'(\xi)|, |Q(\xi)| |P'(\xi)|^2 \} \overline{P}^{n-2} \overline{Q}^{n-1}$$

$$1 \ll \max \{ |Q(\xi)| |P'(\xi)| |Q'(\xi)|, |P(\xi)| |Q'(\xi)|^2 \} \overline{P}^{n-1} \overline{Q}^{n-2}$$

THEOREM 5 (Wirsing, 1961): For any real number $\xi \notin A_n$ there exist infinitely many algebraic numbers $\alpha \in A_n$ with

$$|\xi - \alpha| \ll H(\alpha)^{-\frac{n}{2} - \frac{3}{2}}$$

THEOREM 5 (Wirsing, 1961): For any real number $\xi \notin A_n$ there exist infinitely many algebraic numbers $\alpha \in A_n$ with

$$|\xi - \alpha| \ll H(\alpha)^{-\frac{n}{2} - \frac{3}{2}}$$

THEOREM 6 (Wirsing, 1961): For any real number $\xi \notin A_n$ there exist infinitely many algebraic numbers $\alpha \in A_n$ with

$$|\xi - \alpha| \ll H(\alpha)^{-\frac{n}{2} - \lambda_n}, \quad \lim_{n \rightarrow \infty} \lambda_n = 2$$

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n	Th. 5, 1961	Th. 6, 1961	Th. 7, 1993	Conjecture
3	3	3.28	3.5	4
4	3.5	3.82	4.12	5
5	4	4.35	4.71	6
6	4.5	4.87	5.28	7
7	5	5.39	5.84	8
8	5.5	5.9	6.39	9
9	6	6.41	6.93	10
10	6.5	6.92	7.47	11
15	9	9.44	10.09	16
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4	3.5	3.82	4.12	4.45	5
5	4	4.35	4.71	5.14	6
6	4.5	4.87	5.28	5.76	7
7	5	5.39	5.84	6.36	8
8	5.5	5.9	6.39	6.93	9
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Consider a sequence of polynomials $P_i \in Z[x]$ of degree $\leq n$ such that

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EXAMPLE: Let $n = 1$ and $\xi = \frac{1 + \sqrt{5}}{2}$.

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EXAMPLE: Let $n = 1$ and $\xi = \frac{1 + \sqrt{5}}{2}$. Then

$P_1(x) = x - 2$	$ P_1(\xi) \approx 0.3819$	$\overline{P_1} = 2$
$P_2(x) = 2x - 3$	$ P_2(\xi) \approx 0.2361$	$\overline{P_2} = 3$
$P_3(x) = 3x - 5$	$ P_3(\xi) \approx 0.1459$	$\overline{P_3} = 5$
$P_4(x) = 5x - 8$	$ P_4(\xi) \approx 0.0902$	$\overline{P_4} = 8$
$P_5(x) = 8x - 13$	$ P_5(\xi) \approx 0.0557$	$\overline{P_5} = 13$
$P_6(x) = 13x - 21$	$ P_6(\xi) \approx 0.0344$	$\overline{P_6} = 21$
$P_7(x) = 21x - 34$	$ P_7(\xi) \approx 0.0213$	$\overline{P_7} = 34$

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LEMMA 1: For any $i \geq 1$ we have

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LEMMA: There are infinitely many polynomials $P, Q \in \mathbb{Z}[x]$ of degree $\leq n$, such that

$$|P(\xi)| \ll |P|^{-n}$$

$$|Q(\xi)| \ll |P|^{-n}$$

$$|Q| \ll |P|$$

and

P, Q have no
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Assume to the contrary that there exists a real number $\xi \notin A_n$ such that $|\xi - \alpha| \gg H(\alpha)^{-\omega}$ for any algebraic number $\alpha \in A_n$.

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LEMMA 2: If i is sufficiently large and P_i is irreducible, then

$$1 < |P_i(\xi)|^3 \overline{P_i}^{2\omega+2n-3}$$

LEMMA: Let $P, Q \in Z[x]$ be polynomials of degree d with $1 < d \leq n$. Suppose that P and Q have no common root. Then at least one of the following estimates is true:

$$1 \ll \max \{|P(\xi)|, |Q(\xi)|\}^2 \max \{\overline{P}, \overline{Q}\}^{2n-2}$$

$$1 \ll \max \{|P(\xi)||P'(\xi)||Q'(\xi)|, |Q(\xi)||P'(\xi)|^2\} \overline{P}^{n-2} \overline{Q}^{n-1}$$

$$1 \ll \max \{|Q(\xi)||P'(\xi)||Q'(\xi)|, |P(\xi)||Q'(\xi)|^2\} \overline{P}^{n-1} \overline{Q}^{n-2}$$

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PROOF OF THEOREM 5:

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PROOF OF THEOREM 5:

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PROOF OF THEOREM 5:

$$1 < |P_i(\xi)|^3 \overline{P_i}^{2\omega+2n-3} < 1$$

LEMMA 1: For any $i \geq 1$ we have

$$|P_i(\xi)| < |P_i|^{-n}$$

LEMMA 1: For any $i \geq 1$ we have

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LEMMA 3: We have

$$|\xi - \alpha| \ll \frac{|P_i(\xi)|}{|P'_i(\xi)|}$$

where α is the root of P_i closest to ξ .

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$$|P_i(\xi)| < |P_i|^{-n}$$

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REMARK:

LEMMA 1: For any $i \geq 1$ we have

$$|P_i(\xi)| < \overline{P_i}^{-n}$$

LEMMA 2: If i is sufficiently large and P_i is irreducible, then

$$1 < |P_i(\xi)|^3 \overline{P_i}^{2\omega+2n-3}$$

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REMARK: If $|P'_i(\xi)| \approx \overline{P_i}$

LEMMA 1: For any $i \geq 1$ we have

$$|P_i(\xi)| < \overline{P_i}^{-n}$$

LEMMA 2: If i is sufficiently large and P_i is irreducible, then

$$1 < |P_i(\xi)|^3 \overline{P_i}^{2\omega+2n-3}$$

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$$|\xi - \alpha| \ll \frac{|P_i(\xi)|}{|P_i'(\xi)|}$$

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REMARK: If $|P_i'(\xi)| \approx \overline{P_i}$, then

$$|\xi - \alpha| \ll \frac{|P_i(\xi)|}{\overline{P_i}}$$

LEMMA 1: For any $i \geq 1$ we have

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LEMMA 2: If i is sufficiently large and P_i is irreducible, then

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REMARK: If $|P_i'(\xi)| \approx \overline{P_i}$, then

$$|\xi - \alpha| \ll \frac{|P_i(\xi)|}{\overline{P_i}} \ll \overline{P_i}^{-n-1}$$

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REMARK: If $|P_i'(\xi)| \approx \overline{P_i}$, then

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Consider a sequence of polynomials $P_i \in Z[x]$ of degree $\leq n$ such that

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We have

$$\overline{G} \approx |G'(\xi)|$$

THEOREM: For any real number $\xi \notin A_n$ there exist infinitely many algebraic numbers $\alpha \in A_n$ with

$$|\xi - \alpha| \ll H(\alpha)^{-\omega}$$

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$$|\xi - \alpha| \ll \frac{|P(\xi)|}{|P'(\xi)|} \quad (1)$$

where α is the root of P closest to ξ .

PROOF OF THE THEOREM: Assume to the contrary that there exists a real number $\xi \notin A_n$ such that

$$|\xi - \alpha| \gg H(\alpha)^{-\omega} \quad (2)$$

for any algebraic number $\alpha \in A_n$. From (1) and (2) it follows that

$$|P'(\xi)| \ll |P(\xi)| |P|^\omega \quad (3)$$

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where α is the root of P_i closest to ξ .

EXAMPLE: Consider $P(x) = x^2 - 1000x + 1000$. Then

$$|x^2 - 1000x + 1000| \ll 1000^{-2} \quad \text{at} \quad \xi = 1.001002003\dots$$

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Polynomials

$$P_i(x), \quad P_{i+1}(x), \quad G(x)$$

are linearly independent.

$$P_i(x) = 100x^5 + 100x^4 + 100x^3 + 100x^2 + 100x + 100$$

$$P_{i+1}(x) = 200x^5 + 200x^4 + 200x^3 + 200x^2 + 200x + 200$$

$$P_i(x) = 100x^5 + 100x^4 + 100x^3 + 100x^2 + 100x + 100$$

$$G_1(x) = 190x^5 + 300x^4 + 300x^3 + 300x^2 + 300x + 300$$

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$$P_i(x) = 100x^5 + 100x^4 + 100x^3 + 100x^2 + 100x + 100$$

$$G_2(x) = 190x^5 + 290x^4 + 400x^3 + 400x^2 + 400x + 400$$

$$G_1(x) = 190x^5 + 300x^4 + 300x^3 + 300x^2 + 300x + 300$$

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$$G_2(x) = 190x^5 + 290x^4 + 400x^3 + 400x^2 + 400x + 400$$

$$G_1(x) = 190x^5 + 300x^4 + 300x^3 + 300x^2 + 300x + 300$$

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$$P_i(x) = 100x^5 + 100x^4 + 100x^3 + 100x^2 + 100x + 100$$

Using

$$P_i(x), \quad P_{i+1}(x), \quad G_1(x), \dots, G_{n-2}(x)$$

we construct $L_i(x) \in Z[x]$ of degree $\leq n$ such that

$$|L_i(\xi)| < |P_{i-1}(\xi)|^{\frac{1}{\omega-1}} \overline{|P_{i-1}|}^{-n+1}$$

$$\overline{|L_i|} < |P_{i-1}(\xi)|^{3-\omega-\frac{\omega-2}{\omega-1}} \overline{|P_i|}^{-(n-2)(\omega-1)} \overline{|P_{i-1}|}^{-n+2}$$

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$$1 < |P_i(\xi)| \overline{|P_{i+1}|}^{\frac{2\omega+n-2}{3}(1-\varrho)} \overline{|P_i|}^{\frac{n-1}{3}(1-\varrho)+\Phi(n)\varrho}$$

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we construct $L_i(x) \in Z[x]$ of degree $\leq n$ such that

$$|L_i(\xi)| < |P_{i-1}(\xi)|^{\frac{1}{\omega-1}} \overline{|P_{i-1}|}^{-n+1}$$

$$\overline{|L_i|} < |P_{i-1}(\xi)|^{3-\omega-\frac{\omega-2}{\omega-1}} \overline{|P_i|}^{-(n-2)(\omega-1)} \overline{|P_{i-1}|}^{-n+2}$$

$$\overline{|L_i|} \approx |L'_i(\xi)|$$

$$1 < |P_i(\xi)| \overline{|P_{i+1}|}^{\frac{2\omega+n-2}{3}(1-\varrho)} \overline{|P_i|}^{\frac{n-1}{3}(1-\varrho)+\Phi(n)\varrho}$$

$$\Phi(n) = 2\omega + 2n - 3 - 2(n-1) \frac{\omega-1}{\omega-2}$$

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THEOREM 9: For any real number $\xi \notin A_n$ there exist infinitely many algebraic numbers $\alpha \in A_n$ with

$$|\xi - \alpha| \ll H(\alpha)^{-\omega},$$

where

$$\omega = \frac{n}{2} + \lambda_n, \quad \lim_{n \rightarrow \infty} \lambda_n = 4$$

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$$2x^5 - (n + 12)x^4 + (2n + 30)x^3 + (2n - 41)x^2 - (3n - 29)x + 2n - 10 \quad \text{if } n > 5.$$

n	Th. 5, 1961	Th. 6, 1961	Th. 7, 1993	Th. 9, 2005	Conjecture
3	3	3.28	3.5	3.73	4
4	3.5	3.82	4.12	4.45	5
5	4	4.35	4.71	5.14	6
6	4.5	4.87	5.28	5.76	7
7	5	5.39	5.84	6.36	8
8	5.5	5.9	6.39	6.93	9
9	6	6.41	6.93	7.50	10
10	6.5	6.92	7.47	8.06	11
15	9	9.44	10.09	10.77	16
20	11.5	11.95	12.67	13.40	21
50	26.5	26.98	27.84	28.70	51
100	51.5	51.99	52.92	53.84	101

THEOREM 10 (Davenport - Schmidt, 1968): Let $n \geq 3$. Let ξ be real, but not algebraic of degree ≤ 2 . Then there are infinitely many algebraic integers α of degree ≤ 3 which satisfy

$$0 < |\xi - \alpha| \ll H(\alpha)^{-\eta}, \quad \eta = \frac{1}{2}(3 + \sqrt{5}) = 2.618\dots$$

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