

On approximation of real numbers by algebraic
numbers of bounded degree

BY

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THEOREM 1 (Dirichlet, 1842). For any real irrational number ξ there exist infinitely many rational numbers p/q such that

$$\left| \xi - \frac{p}{q} \right| < q^{-2}.$$

\Downarrow

$$|q\xi - p| < q^{-1}.$$

THEOREM 2. For any real irrational number ξ there exist infinitely many polynomials $P \in \mathbb{Z}[x]$ of the first degree such that

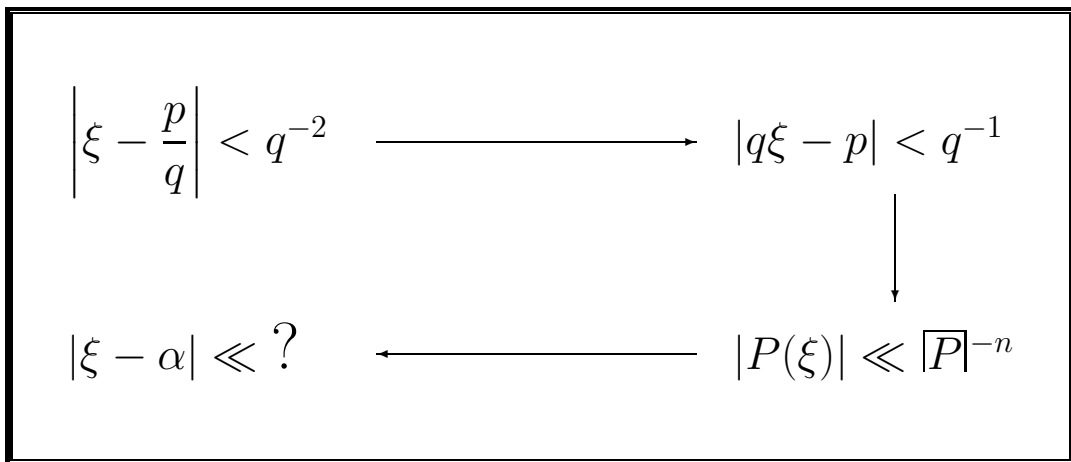
$$|P(\xi)| \ll \overline{P}^{-1},$$

where \overline{P} denotes the height of the polynomial P , that is the maximum of absolute values of its coefficients, \ll is the Vinogradov symbol.

THEOREM 3. For any real number $\xi \notin \mathbb{A}_n$ there exist infinitely many polynomials $P \in \mathbb{Z}[x]$ of degree $\leq n$ such that

$$|P(\xi)| \ll \overline{P}^{-n},$$

where \mathbb{A}_n is the set of real algebraic numbers of degree $\leq n$.



CONJECTURE (WIRSING, 1961). For any real number $\xi \notin \mathbb{A}_n$, there exist infinitely many algebraic numbers $\alpha \in \mathbb{A}_n$ with

$$|\xi - \alpha| \ll H(\alpha)^{-n-1+\epsilon}, \quad \epsilon > 0,$$

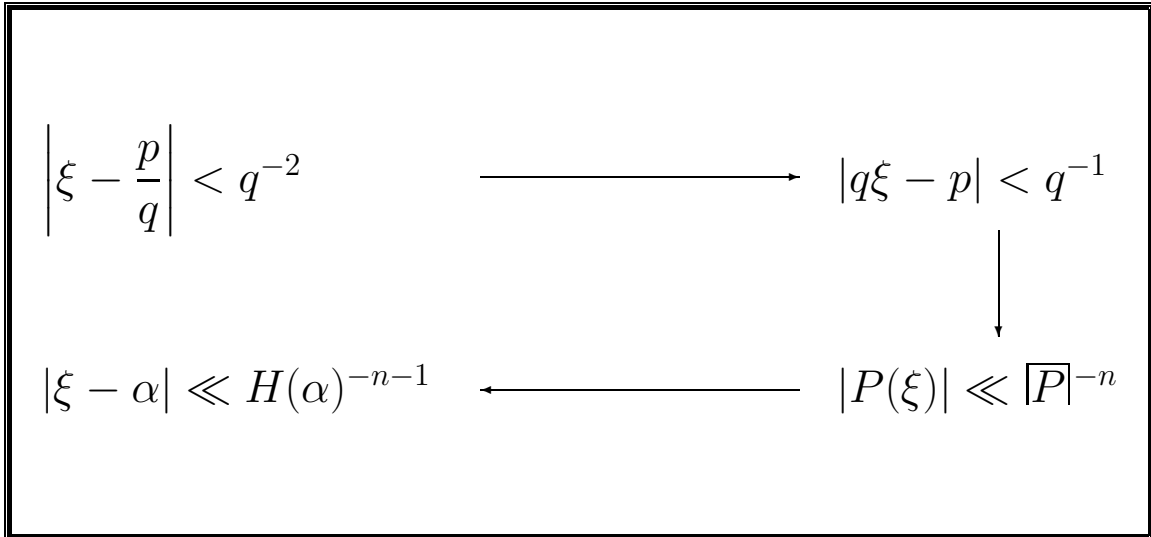
where $H(\alpha)$ is the height of α .

Further W. M. Schmidt conjectured that the exponent

$$-n - 1 + \epsilon$$

can be replaced even by

$$-n - 1.$$



At the moment this Conjecture is proved only for

$$n = 1 \quad \Rightarrow \quad |\xi - \alpha| \ll H(\alpha)^{-2} \quad (\text{Dirichlet, 1842})$$

$$n = 2 \quad \Rightarrow \quad |\xi - \alpha| \ll H(\alpha)^{-3} \quad (\text{Davenport - Schmidt, 1967})$$

$$n > 2 \quad \Rightarrow \quad ???$$

Consider the polynomial

$$P(x) = a_n x^n + \dots + a_1 x + a_0 = a_n (x - \alpha_1) \cdot \dots \cdot (x - \alpha_n).$$

Without loss of generality we can assume that α_1 is the root of $P(x)$ closest to ξ . We have

$$|\xi - \alpha_1| \ll \frac{|P(\xi)|}{|P'(\xi)|}.$$

By Theorem 3 there are infinitely many polynomials $P \in \mathbb{Z}[x]$ of degree $\leq n$ such that

$$|P(\xi)| \ll \overline{P}^{-n}.$$

Let $n = 1$. Then

$$|P'(\xi)| = |a_1| \asymp \overline{P} \quad \Rightarrow \quad |\xi - \alpha_1| \ll \frac{\overline{P}^{-1}}{\overline{P}} = \overline{P}^{-2}$$

Let $n = 2$. Then for some $\delta \leq 1$

$$|P'(\xi)| = |2a_2\xi + a_1| \asymp \overline{P}^\delta \quad \Rightarrow \quad |\xi - \alpha_1| \ll \frac{\overline{P}^{-2}}{\overline{P}^\delta} = \overline{P}^{-2-\delta} ???$$

QUESTION: Can one prove that the following is impossible:

All polynomials with $|P(\xi)| \ll \overline{P}^{-n}$ have
a “small” derivative $|P'(\xi)| \asymp \overline{P}^\delta$, $\delta < 1$.

ANSWER: Yes, for $n=1$ (Dirichlet, 1842)

$n=2$ (Davenport – Schmidt, 1967)

THEOREM 4 (Wirsing, 1961). For any real number $\xi \notin \mathbb{A}_n$ there exist infinitely many algebraic numbers $\alpha \in \mathbb{A}_n$ with

$$|\xi - \alpha| \ll H(\alpha)^{-\frac{n}{2} - \gamma_n}, \quad \lim_{n \rightarrow \infty} \gamma_n = 2.$$

By Dirichlet's Box principle there are infinitely polynomials

$$P(x) = a_n(x - \alpha_1) \cdot \dots \cdot (x - \alpha_n)$$

such that

$$|P(\xi)| \ll |\overline{P}|^{-n}, \quad (*)$$

therefore

$$|\xi - \alpha_1| \cdot \dots \cdot |\xi - \alpha_n| \ll |\overline{P}|^{-n} a_n^{-1}.$$

Even if $a_n = |\overline{P}|$, we can only prove that

$$|\xi - \alpha_1| \cdot \dots \cdot |\xi - \alpha_n| \ll |\overline{P}|^{-n-1} \ll H(\alpha_1)^{-n-1},$$

↓

$$|\xi - \alpha_1| \ll H(\alpha_1)^{-\frac{n+1}{n}} ???$$

It is also clear, that the worse case for us is when

$$|\xi - \alpha_1| = \dots = |\xi - \alpha_n|. \quad (**)$$

QUESTION: Can one prove that for infinitely many polynomials $P \in \mathbb{Z}[x]$ with (*) the situation (**) is impossible?

ANSWER: For infinitely many polynomials $P \in \mathbb{Z}[x]$ with (*) we have:

$$|\xi - \alpha_1| \ll |\xi - \alpha_2| \ll 1,$$

$|\xi - \alpha_3|, \dots, |\xi - \alpha_n|$ are "big".

THEOREM 5 (Bernik-Tsishchanka, 1993). For any real number $\xi \notin \mathbb{A}_n$ there exist infinitely many algebraic numbers $\alpha \in \mathbb{A}_n$ with

$$|\xi - \alpha| \ll H(\alpha)^{-\frac{n}{2} - \gamma_n}, \quad \lim_{n \rightarrow \infty} \gamma_n = 3.$$

THEOREM 6. For any real number $\xi \notin \mathbb{A}_n$ there exist infinitely many algebraic numbers $\alpha \in \mathbb{A}_n$ with

$$|\xi - \alpha| \ll H(\alpha)^{-\frac{n}{2} - \gamma_n}, \quad \lim_{n \rightarrow \infty} \gamma_n = 4.$$

The following table contains the values of

$$\frac{n}{2} + \gamma_n$$

corresponding to Wirsing's Theorem (1961), the Theorem of Bernik-Tsishchanka (1993), Theorem 6 (2001), and the Conjecture (Conj.):

n	1961	1993	2001	Conj.
3	3.28	3.5	3.73	4
4	3.82	4.12	4.45	5
5	4.35	4.71	5.14	6
6	4.87	5.28	5.76	7
7	5.39	5.84	6.36	8
8	5.9	6.39	6.93	9
9	6.41	6.93	7.50	10
10	6.92	7.47	8.06	11
15	9.44	10.09	10.77	16
20	11.95	12.67	13.40	21
50	26.98	27.84	28.70	51
100	51.99	52.92	53.84	101

OPEN QUESTION. For any real number $\xi \notin \mathbb{A}_n$, there exist infinitely many algebraic numbers $\alpha \in \mathbb{A}_n$ with

$$|\xi - \alpha| \ll H(\alpha)^{-\frac{n}{2} - \gamma_n}, \quad \lim_{n \rightarrow \infty} \gamma_n = \infty.$$

Consider the sequence of polynomials $P_i \in \mathbb{Z}[x]$ of degree $\leq n$ such that

$$(i) \quad |P_1(\xi)| > |P_2(\xi)| > \dots > |P_i(\xi)| > \dots,$$

$$(ii) \quad \overline{P_1} < \overline{P_2} < \dots < \overline{P_i} < \dots,$$

(iii) for any $i > 1$ the polynomial P_i has the smallest height among other polynomials P with $|P(\xi)| < |P_{i-1}(\xi)|$.

LEMMA. For any $i \geq 1$ we have:

$$|P_i(\xi)| \ll \overline{P_{i+1}}^{-n}.$$

THEOREM 7. Let $n = 3$. Suppose that for any $i \geq 1$ the following estimation holds:

$$|P_i(\xi)| \ll \overline{P_{i+1}}^{-1} \overline{P_{i+2}}^{-2}.$$

Then the Conjecture is true, that is for any real number $\xi \notin \mathbb{A}_3$ there exist infinitely many algebraic numbers $\alpha \in \mathbb{A}_3$ such that

$$|\xi - \alpha| \ll H(\alpha)^{-4}.$$

THEOREM 8. For any real number $\xi \notin \mathbb{A}_3$ there exist infinitely many algebraic numbers $\alpha \in \mathbb{A}_3$ such that

$$|\xi - \alpha| \ll H(\alpha)^{-3.7475}.$$

OPEN QUESTION. Let $n = 3$. Suppose that for any $i \geq 1$ the following estimation holds:

$$|P_i(\xi)| \ll \overline{P_{i+1}}^{-2} \overline{P_{i+2}}^{-1}.$$

Prove that the Conjecture is true.

CONJECTURE. For any two real numbers $\xi_1, \xi_2 \notin \mathbb{A}_n$ there exist infinitely many algebraic numbers α_1, α_2 such that

$$\begin{cases} |\xi_1 - \alpha_1| \ll \overline{P}^{-(n+1)/2}, \\ |\xi_2 - \alpha_2| \ll \overline{P}^{-(n+1)/2}, \end{cases}$$

where $P(x) \in \mathbb{Z}[x]$, $P(\alpha_1) = P(\alpha_2) = 0$, $\deg P \leq n$. The implicit constant in \ll should depend on ξ_1, ξ_2 , and n .

One can prove it for almost all real numbers. However, this is not true for all real numbers. In 2001 D. Roy and M. Waldschmidt constructed the following

COUNTER-EXAMPLE (Roy-Waldschmidt, 2001). For any $\epsilon > 0$ and any sufficiently large n there exist real numbers ξ_1 and ξ_2 such that

$$\max\{|\xi_1 - \alpha_1|, |\xi_2 - \alpha_2|\} > \overline{P}^{-\sqrt{n^{1+6\epsilon}}}.$$

At the same time we can prove the following

THEOREM 9. For any real numbers $\xi_1, \xi_2 \notin \mathbb{A}_n$ at least one of the following assertions is true:

(i) There exist infinitely many algebraic numbers α_1, α_2 of degree $\leq n$ such that

$$\begin{cases} |\xi_1 - \alpha_1| \ll \overline{P}^{-\frac{n}{8} - \frac{3}{8}}, \\ |\xi_2 - \alpha_2| \ll \overline{P}^{-\frac{n}{8} - \frac{3}{8}}. \end{cases}$$

(ii) For some $\xi \in \{\xi_1, \xi_2\}$ there exist infinitely many algebraic numbers α of degree $2 \leq k \leq \frac{n+2}{4}$ such that

$$|\xi - \alpha| \ll H(\alpha)^{-\frac{n+2}{4}-1}.$$