On approximation of real, complex, and p-adic numbers by algebraic numbers of bounded degree

BY

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I. On approximation by rational numbers

THEOREM 1 (Dirichlet, 1842). For any real irrational number ξ there exist infinitely many rational numbers p/q such that

$$\left|\xi - \frac{p}{q}\right| < \frac{1}{q^2}.$$

Example: Let $\xi = e$. Consider the continued fraction expansion:

$$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \dots}}}}}$$

We have

$$2 + \frac{1}{1} = 3 \qquad 2 + \cfrac{1}{1 + \cfrac{1}{2}} = \frac{8}{3} \qquad 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1}}} = \frac{11}{4}$$

$$2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1}}}} = \frac{19}{7} \qquad 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4}}}}} = \frac{87}{32}$$

$$\delta_0 = 2 \qquad |e - 2| < 1$$

$$\delta_1 = 3 \qquad |e - 3| < 1$$

$$\delta_2 = \frac{8}{3} \qquad \left|e - \frac{8}{3}\right| < \frac{1}{3^2}$$

$$\delta_3 = \frac{11}{4} \qquad \left|e - \frac{11}{4}\right| < \frac{1}{4^2}$$

$$\delta_4 = \frac{19}{7} \qquad \left|e - \frac{19}{7}\right| < \frac{1}{7^2}$$

$$\delta_5 = \frac{87}{32} \qquad \left|e - \frac{87}{32}\right| < \frac{1}{32^2}$$

$$\delta_6 = \frac{106}{39} \qquad \left|e - \frac{106}{39}\right| < \frac{1}{39^2}$$

$$\delta_7 = \frac{193}{71} \qquad \left|e - \frac{193}{71}\right| < \frac{1}{71^2}$$

$$\delta_8 = \frac{1264}{465} \qquad \left|e - \frac{1264}{465}\right| < \frac{1}{465^2}$$

$$\delta_9 = \frac{1457}{536} \qquad \left|e - \frac{1457}{536}\right| < \frac{1}{536^2}$$

$$\delta_{10} = \frac{2721}{1001} \qquad \left|e - \frac{2721}{1001}\right| < \frac{1}{1001^2}$$

$$\delta_0 = 2 \qquad |e-2| < 1$$

$$\boldsymbol{\delta_1 = 3} \qquad \boldsymbol{|e-3| < \frac{1}{2\cdot 1^2}}$$

$$\boldsymbol{\delta_2 = \frac{8}{3}} \qquad \boldsymbol{\left|e-\frac{8}{3}\right| < \frac{1}{2\cdot 3^2}}$$

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$$\boldsymbol{\delta_7 = \frac{193}{71}} \qquad \boldsymbol{\left|e-\frac{193}{71}\right| < \frac{1}{2\cdot 71^2}}$$

$$\boldsymbol{\delta_8 = \frac{1264}{465}} \qquad \boldsymbol{\left|e-\frac{1264}{465}\right| < \frac{1}{2\cdot 465^2}}$$

$$\delta_9 = \frac{1457}{536} \qquad \left|e-\frac{1457}{536}\right| < \frac{1}{536^2}$$

$$\boldsymbol{\delta_{10} = \frac{2721}{1001}} \qquad \boldsymbol{\left|e-\frac{2721}{1001}\right| < \frac{1}{2\cdot 1001^2}}$$

THEOREM 2. For any real irrational number ξ there exist infinitely many rational numbers p/q such that

$$\left|\xi - \frac{p}{q}\right| < \frac{1}{2q^2}.$$

$$\delta_0 = 2 \qquad |e - 2| < 1$$

$$\boldsymbol{\delta_1 = 3 \qquad |e - 3| < \frac{1}{\sqrt{5} \cdot 1^2}}$$

$$\boldsymbol{\delta_2 = \frac{8}{3} \qquad \left|e - \frac{8}{3}\right| < \frac{1}{2 \cdot 3^2}}$$

$$\delta_3 = \frac{11}{4} \qquad \left|e - \frac{11}{4}\right| < \frac{1}{4^2}$$

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$$\boldsymbol{\delta_{10} = \frac{2721}{1001} \qquad \left|e - \frac{2721}{1001}\right| < \frac{1}{\sqrt{5} \cdot 1001^2}}$$

THEOREM 3 (Hurwitz). For any real irrational number ξ there exist infinitely many rational numbers p/q such that

$$\left|\xi - \frac{p}{q}\right| < \frac{1}{\sqrt{5}q^2}.$$

This result is the best possible.

Example:

$$\xi = \frac{1+\sqrt{5}}{2} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \dots}}}}}$$

II. Polynomial Interpretation

For any real irrational number ξ there exist infinitely many rational numbers p/q such that

$$\left|\xi - \frac{p}{q}\right| < \frac{1}{q^2}.$$

$$\Downarrow$$

$$|q\xi - p| < q^{-1}.$$

THEOREM 4. For any real irrational number ξ there exist infinitely many polynomials $P \in \mathbb{Z}[x]$ of the first degree such that

$$|P(\xi)| \ll \lceil P \rceil^{-1},$$

where $\lceil P \rceil$ denotes the height of the polynomial P, that is the maximum of absolute values of its coefficients, $\ll$ is the Vinogradov symbol.

THEOREM 5. For any real number $\xi \notin \mathbb{A}_n$ there exist infinitely many polynomials $P \in \mathbb{Z}[x]$ of degree $\leq n$ such that

$$|P(\xi)| \ll \lceil P \rceil^{-n},$$

where $\mathbb{A}_n$ is the set of real algebraic numbers of degree $\leq n$.

$$\begin{array}{ccc}
\left|\xi - \dfrac{p}{q}\right| < q^{-2} & \longrightarrow & |q\xi - p| < q^{-1} \\
 & & \downarrow \\
|\xi - \alpha| \ll\ ? & \longleftarrow & |P(\xi)| \ll \lceil P \rceil^{-n}
\end{array}$$

III. Conjecture of Wirsing

Conjecture (Wirsing, 1961). For any real number $\xi \notin \mathbb{A}_n$, there exist infinitely many algebraic numbers $\alpha \in \mathbb{A}_n$ with

$$|\xi - \alpha| \ll H(\alpha)^{-n-1+\epsilon}, \quad \epsilon > 0,$$

where $H(\alpha)$ is the height of α.

Further W. M. Schmidt conjectured that the exponent

$$-n-1+\epsilon$$

can be replaced even by

$$-n-1.$$

$$\begin{array}{ccc} \left|\xi - \dfrac{p}{q}\right| < q^{-2} & \longrightarrow & |q\xi - p| < q^{-1} \\ & & \downarrow \\ |\xi - \alpha| \ll H(\alpha)^{-n-1} & \longleftarrow & |P(\xi)| \ll \lceil P \rceil^{-n} \end{array}$$

At the moment this Conjecture is proved only for

$n = 1 \quad \Rightarrow \quad |\xi - \alpha| \ll H(\alpha)^{-2}$ (Dirichlet, 1842)

$n = 2 \quad \Rightarrow \quad |\xi - \alpha| \ll H(\alpha)^{-3}$ (Davenport – Schmidt, 1967)

$n > 2 \quad \Rightarrow \quad$???

Consider the polynomial

$$P(x) = a_n x^n + \ldots + a_1 x + a_0 = a_n(x - \alpha_1) \cdot \ldots \cdot (x - \alpha_n).$$

Without loss of generality we can assume that α_1 is the root of $P(x)$ closest to ξ. We have

$$|\xi - \alpha_1| \ll \frac{|P(\xi)|}{|P'(\xi)|}.$$

By Theorem 3 there are infinitely many polynomials $P \in \mathbb{Z}[x]$ of degree $\leq n$ such that

$$|P(\xi)| \ll \lceil P \rceil^{-n}.$$

Let $n = 1$. Then

$$|P'(\xi)| = |a_1| \asymp \lceil P \rceil \quad \Rightarrow \quad \boxed{|\xi - \alpha_1| \ll \frac{\lceil P \rceil^{-1}}{\lceil P \rceil} = \lceil P \rceil^{-2}}$$

Let $n = 2$. Then for some $\delta \leq 1$

$$|P'(\xi)| = |2a_2\xi + a_1| \asymp \lceil P \rceil^{\delta} \quad \Rightarrow \quad \boxed{|\xi - \alpha_1| \ll \frac{\lceil P \rceil^{-2}}{\lceil P \rceil^{\delta}} = \lceil P \rceil^{-2-\delta} \ ???}$$

QUESTION: Can one prove that the following is impossible:

All polynomials with $|P(\xi)| \ll \lceil P \rceil^{-n}$ have

a "small" derivative $|P'(\xi)| \asymp \lceil P \rceil^{\delta}, \ \delta < 1.$

ANSWER: Yes, for n=1 (Dirichlet, 1842)

n=2 (Davenport – Schmidt, 1967)

IV. Theorem of Wirsing

THEOREM 6 (Wirsing, 1961). For any real number $\xi \notin \mathbb{A}_n$ there exist infinitely many algebraic numbers $\alpha \in \mathbb{A}_n$ with

$$|\xi - \alpha| \ll H(\alpha)^{-\frac{n}{2}-\lambda_n}, \quad \lim_{n\to\infty} \lambda_n = 2.$$

By Dirichlet's Box principle there are infinitely polynomials

$$P(x) = a_n(x-\alpha_1)\cdot \ldots \cdot (x-\alpha_n)$$

such that

$$|P(\xi)| \ll \lceil P \rceil^{-n},$$

therefore

$$|\xi - \alpha_1| \cdot \ldots \cdot |\xi - \alpha_n| \ll \lceil P \rceil^{-n} a_n^{-1}.$$

Even if $a_n = \lceil P \rceil$, we can only prove that

$$|\xi - \alpha_1| \cdot \ldots \cdot |\xi - \alpha_n| \ll \lceil P \rceil^{-n-1} \ll H(\alpha_1)^{-n-1},$$

$$\Downarrow$$

$$\boxed{|\xi - \alpha_1| \ll H(\alpha_1)^{-\frac{n+1}{n}} \text{ ???}}$$

It is also clear, that the worth case for us is when

$$|\xi - \alpha_1| = \ldots = |\xi - \alpha_n|.$$

QUESTION: Can one prove that for infinitely many polynomials $P \in \mathbb{Z}[x]$ with $|P(\xi)| \ll \lceil P \rceil^{-n}$ the situation

$$|\xi - \alpha_1| = \ldots = |\xi - \alpha_n|$$

is impossible?

ANSWER: For infinitely many polynomials $P \in \mathbb{Z}[x]$ with $|P(\xi)| \ll \lceil P \rceil^{-n}$ we have:

$$|\xi - \alpha_1| \ll |\xi - \alpha_2| \ll 1,$$

$$|\xi - \alpha_3|, \ldots, |\xi - \alpha_n| \text{ are "big".}$$

Step 1: Construct ∞-many $P,\ Q \in \mathbb{Z}[x],\ \deg\ P, Q \le n$, such that

$$\boxed{\begin{aligned} &|P(\xi)| \ll \lceil P \rceil^{-n} \\ &|Q(\xi)| \ll \lceil Q \rceil^{-n} \\ &\lceil P \rceil \ll \lceil Q \rceil \end{aligned}} \quad \text{and} \quad \boxed{\begin{array}{c} P, Q \ \text{ have no} \\ \text{common root} \end{array}}$$

Step 2. Consider the resultant of P, Q :

$$R(P, Q) = a_m^{\ell} b_{\ell}^{m} \prod_{1 \le i, j \le n} (\alpha_i - \beta_j).$$

On the one hand,

$$R(P, Q) \neq 0,$$

since P, Q have no common root. Moreover,

$$R(P, Q) \in \mathbb{Z},$$

since P, Q have integer coefficients.

Therefore we get

$$|R(P, Q)| \ge 1.$$

Step 3. On the other hand,

$$\begin{aligned} |R(P,Q)| &= a_m^{\ell} b_{\ell}^{m} \prod_{1\le i,j\le n} |\alpha_i - \beta_j| \\ &\ll \lceil P \rceil^{2n} \prod_{1\le i,j\le n} |\alpha_i - \beta_j| \\ &\ll \lceil P \rceil^{2n} \prod_{1\le i,j\le n} \max(|\xi - \alpha_i|,\ |\xi - \beta_j|). \end{aligned}$$

If

$$|\xi - \alpha_1| = \ldots = |\xi - \alpha_n| \ll \lceil P \rceil^{-1-\frac{1}{n}},$$

$$|\xi - \beta_1| = \ldots = |\xi - \beta_n| \ll \lceil P \rceil^{-1-\frac{1}{n}},$$

then

$$|R(P,Q)| \ll \lceil P \rceil^{2n} \lceil P \rceil^{\left(-1-\frac{1}{n}\right)n^2} = \lceil P \rceil^{n-n^2} < 1,$$

which contradicts to Step 2.

Lemma (Wirsing, 1961):

$$|\xi - \gamma| \ll \max \begin{cases} |P(\xi)|^{\frac{1}{2}}|Q(\xi)|\lceil P\rceil^{n-\frac{3}{2}}, \\ \\ |P(\xi)||Q(\xi)|^{\frac{1}{2}}\lceil P\rceil^{n-\frac{3}{2}}, \end{cases}$$

where γ is a root of P or Q closest to ξ.

Since

$$|P(\xi)| \ll \lceil P\rceil^{-n},$$

$$|Q(\xi)| \ll \lceil Q\rceil^{-n},$$

we get

$$|\xi - \gamma| \ll \lceil P\rceil^{-\frac{n}{2}-n+n-\frac{3}{2}} = \lceil P\rceil^{-\frac{n}{2}-\frac{3}{2}}.$$

V. "Big Derivative" Method

THEOREM 7 (Bernik-Tsishchanka, 1993). For any real number $\xi \notin \mathbb{A}_n$ there exist infinitely many algebraic numbers $\alpha \in \mathbb{A}_n$ with

$$|\xi - \alpha| \ll H(\alpha)^{-\frac{n}{2}-\lambda_n}, \quad \lim_{n\to\infty} \lambda_n = 3.$$

The following table contains the values of

$$\frac{n}{2} + \lambda_n$$

corresponding to Wirsing's Theorem (1961), the Theorem of Bernik-Tsishchanka (1993), and the Conjecture:

n	1961	1993	Conj.
3	3.28	3.5	4
4	3.82	4.12	5
5	4.35	4.71	6
6	4.87	5.28	7
7	5.39	5.84	8
8	5.9	6.39	9
9	6.41	6.93	10
10	6.92	7.47	11
15	9.44	10.09	16
20	11.95	12.67	21
50	26.98	27.84	51
100	51.99	52.92	101

Fix some $H > 0$. By Dirichlet's Box Principle there exists an integer polynomial P such that

$$\begin{aligned} &|a_n| \ll H,\ \ldots,\ |a_2| \ll H, \quad |a_1| \ll H^{1+\epsilon}, \quad |a_0| \ll H^{1+\epsilon}, \\ &|P(\xi)| \ll H^{-n-\epsilon}, \end{aligned} \tag{1}$$

where $\epsilon > 0$. We now consider two cases:

Case A: Let

$$\max(|a_1|,\ |a_0|) \gg H,$$

that is

$$\max(|a_1|,\ |a_0|) = H^{1+\delta} = \lceil P \rceil, \qquad 0 < \delta \leq \epsilon.$$

It is clear that in this case the derivative of P is "big", that is

$$|P'(\xi)| \asymp H^{1+\delta}. \tag{2}$$

We have the following well-known inequality

$$|\xi - \alpha| \ll \frac{|P(\xi)|}{|P'(\xi)|}, \tag{3}$$

where α is the root of the polynomial P closest to ξ. Substituting (1) and (2) into (3), we get

$$|\xi - \alpha| \ll \frac{H^{-n-\epsilon}}{H^{1+\delta}} = H^{-(1+\delta)\frac{n+1+\epsilon+\delta}{1+\delta}} = \lceil P \rceil^{-\frac{n+1+\epsilon+\delta}{1+\delta}} \ll H(\alpha)^{-\frac{n+1+2\epsilon}{1+\epsilon}}.$$

Case B: Let

$$\max(|a_1|,\ |a_0|) \ll H,$$

then

$$\overline{P} \ll H. \tag{4}$$

Using Dirichlet's Box we construct an integer polynomial Q such that

$$\begin{aligned} &|b_n| \ll H,\ \ldots,\ |b_2| \ll H, \quad |b_1| \ll H^{1+\epsilon}, \quad |b_0| \ll H^{1+\epsilon}, \\ &|Q(\xi)| \ll H^{-n-\epsilon}, \end{aligned} \tag{5}$$

If $\max(|b_1|,\ |b_0|) \gg H$, then

$$|\xi - \beta| \ll H(\beta)^{-\frac{n+1+2\epsilon}{1+\epsilon}}.$$

If $\max(|b_1|,\ |b_0|) \ll H$, then

$$\overline{Q} \ll H. \tag{6}$$

Then we can apply Wirsing's Lemma:

$$|\xi - \gamma| \ll \max \begin{cases} |P(\xi)|^{\frac{1}{2}}|Q(\xi)|\overline{P}^{n-\frac{3}{2}}, \\ \\ |P(\xi)||Q(\xi)|^{\frac{1}{2}}\overline{P}^{n-\frac{3}{2}}, \end{cases}$$

Substituting (4), (5), (6), and $|P(\xi)| \ll H^{-n-\epsilon}$, we get:

$$|\xi - \gamma| \ll H^{-\frac{n}{2}-\frac{3}{2}-\frac{3}{2}\epsilon} \ll H(\gamma)^{-\frac{n}{2}-\frac{3}{2}-\frac{3}{2}\epsilon}.$$

Let us compare estimates in the Case A and Case B:

$$\text{Case A:} \quad |\xi - \alpha| \ll H(\alpha)^{-\frac{n+1+2\epsilon}{1+\epsilon}}$$

$$\text{Case B:} \quad |\xi - \alpha| \ll H(\alpha)^{-\frac{n}{2}-\frac{3}{2}-\frac{3}{2}\epsilon}$$

If we take $\epsilon = 0$, then

$$\text{Case A:} \quad |\xi - \alpha| \ll H(\alpha)^{-n-1}$$

$$\text{Case B:} \quad |\xi - \alpha| \ll H(\alpha)^{-\frac{n}{2}-\frac{3}{2}}$$

On the other hand, if we take $\epsilon = 2$, then

$$\text{Case A:} \quad |\xi - \alpha| \ll H(\alpha)^{-\frac{n+5}{3}}$$

$$\text{Case B:} \quad |\xi - \alpha| \ll H(\alpha)^{-\frac{n}{2}-4.5}$$

Finally, if we choose an optimal value of ϵ, namely

$$\epsilon = 1 - \frac{6}{n},$$

we obtain

$$|\xi - \alpha| \ll H(\alpha)^{-n/2+\lambda_n}, \quad \lim_{n\to\infty} \lambda_n = 3,$$

in both cases.

VI. "Improvement"

Let us consider an integer polynomial P such that

$$|a_n| \ll H, \ \ldots, \ |a_2| \ll H^{1+\epsilon}, \quad |a_1| \ll H^{1+\epsilon}, \quad |a_0| \ll H^{1+\epsilon},$$

$$|P(\xi)| \ll H^{-n-2\epsilon}.$$

We have

$$\text{Case A:} \quad |\xi - \alpha| \ll H(\alpha)^{-\frac{n+1+3\epsilon}{1+\epsilon}}$$

$$\text{Case B:} \quad |\xi - \alpha| \ll H(\alpha)^{-\frac{n}{2}-\frac{3}{2}-3\epsilon}.$$

Put

$$\epsilon = 1 - \frac{10}{n},$$

then

$$|\xi - \alpha| \ll H(\alpha)^{-n/2+\lambda_n}, \quad \lim_{n\to\infty} \lambda_n = 4.5,$$

in both cases.

However, the Case A does not work. In fact,

$$\max(|a_2|, \ |a_1|, \ |a_0|) \gg H \quad \not\Rightarrow \quad |P'(\xi)| \quad \text{is "big"}.$$

VII. Method of "Polynomial Staircase"

In 1996 a new approach to this problem was introduced:

Step 1. Let $R^{(k)}$ be a polynomial with k "big" coefficients. We construct the following n polynomials

$$Q^{(3)}, \dots, Q^{(n+1)},\ P^{(n+1)},$$

which are small at ξ.

Step 2. We prove that they are linearly independent.

Step 3. Using a linear combination of these polynomials, we construct the polynomial

$$L^{(2)} = d_1 Q^{(3)} + \dots + d_{n-1} Q^{(n+1)} + d_n P^{(n+1)}$$

with two "big" coefficients. The Case A does work for L. Moreover, it is possible to show that an influence of the numbers $d_1, \dots, d_n$ is very weak, so

$$|L(\xi)| \ll H^{-n-2\epsilon}.$$

This method allows us to prove the following

THEOREM 8. For any real number $\xi \notin \mathbb{A}_n$ there exist infinitely many algebraic numbers $\alpha \in \mathbb{A}_n$ with

$$|\xi - \alpha| \ll H(\alpha)^{-\frac{n}{2}-\lambda_n}, \quad \lim_{n\to\infty} \lambda_n = 4.$$

The following table contains the values of

$$\frac{n}{2} + \lambda_n$$

corresponding to Wirsing's Theorem (1961), the Theorem of Bernik-Tsishchanka (1993), Theorem 8 (2001), and the Conjecture:

n	1961	1993	2001	Conj.
3	3.28	3.5	3.73	4
4	3.82	4.12	4.45	5
5	4.35	4.71	5.14	6
6	4.87	5.28	5.76	7
7	5.39	5.84	6.36	8
8	5.9	6.39	6.93	9
9	6.41	6.93	7.50	10
10	6.92	7.47	8.06	11
15	9.44	10.09	10.77	16
20	11.95	12.67	13.40	21
50	26.98	27.84	28.70	51
100	51.99	52.92	53.84	101

VIII. Complex case

THEOREM 9 (Wirsing, 1961). For any complex number $\xi \notin \mathbb{A}_n$ there exist infinitely many algebraic numbers $\alpha \in \mathbb{A}_n$ with

$$|\xi - \alpha| \ll H(\alpha)^{-A},$$

where

$$A = \frac{n}{4} + 1.$$

Method: "Resultant"

In 2000 this result was slightly improved:

$$A = \frac{n}{4} + \lambda_n, \quad \text{where} \quad \lim_{n\to\infty} \lambda_n = \frac{3}{2}.$$

Method: "Big Derivative".

Method "Polynomial Staircase": ? ? ?

IX. *P*-adic case

THEOREM 10 (Morrison, 1978). Let $\xi \in \mathbb{Q}_p$. If $\xi \notin \mathbb{A}_n$, then there are infinitely many algebraic numbers $\alpha \in \mathbb{A}_n$ with

$$|\xi - \alpha| \ll H(\alpha)^{-A},$$

where

$$A = \begin{cases} 1+\sqrt{3} & \text{when} \quad n = 2, \\ \dfrac{n}{2} + \dfrac{3}{2} & \text{when} \quad n > 2. \end{cases}$$

THEOREM 11 (Teulié, 2002). If $\xi \notin \mathbb{A}_2$, then there are infinitely many algebraic numbers $\alpha \in \mathbb{A}_2$ with

$$|\xi - \alpha| \ll H(\alpha)^{-3}.$$

The second part of Morrison's theorem was also improved:

$$A = \frac{n}{2} + \lambda_n, \quad \text{where} \quad \lim_{n\to\infty} \lambda_n = 3.$$

Method: "Big Derivative".

Method "Polynomial Staircase": ? ? ?

X. Two Counter-Examples

1. Simultaneous case.

CONJECTURE. For any two real numbers $\xi_1,\ \xi_2 \notin \mathbb{A}_n$ there exist infinitely many algebraic numbers $\alpha_1,\ \alpha_2$ such that

$$\begin{cases} |\xi_1 - \alpha_1| \ll \lceil P \rceil^{-(n+1)/2}, \\ |\xi_2 - \alpha_2| \ll \lceil P \rceil^{-(n+1)/2}, \end{cases}$$

where $P(x) \in \mathbb{Z}[x],\ \ P(\alpha_1) = P(\alpha_2) = 0$, $\deg P \leq n$. The implicit constant in $\ll$ should depend on $\xi_1,\ \xi_2$, and n.

COUNTER-EXAMPLE (Roy-Waldschmidt, 2001). For any sufficiently large n there exist real numbers ξ_1 and ξ_2 such that

$$\max\{|\xi_1 - \alpha_1|, |\xi_2 - \alpha_2|\} > \lceil P \rceil^{-3\sqrt{n}}.$$

THEOREM 12. For any real numbers $\xi_1,\ \xi_2 \notin \mathbb{A}_n$ at least one of the following assertions is true:

(i) There exist infinitely many algebraic numbers $\alpha_1,\ \alpha_2$ of degree $\leq n$ such that

$$\begin{cases} |\xi_1 - \alpha_1| \ll \lceil P \rceil^{-\frac{n}{8} - \frac{3}{8}}, \\ |\xi_2 - \alpha_2| \ll \lceil P \rceil^{-\frac{n}{8} - \frac{3}{8}}. \end{cases}$$

(ii) For some $\xi \in \{\xi_1,\ \xi_2\}$ there exist infinitely many algebraic numbers α of degree $2 \leq k \leq \frac{n+2}{4}$ such that

$$|\xi - \alpha| \ll H(\alpha)^{-\frac{n+2}{4} - 1}.$$

2. Approximation by algebraic integers.

Theorem 13. (Davenport - Schmidt, 1968) Let $n \geq 3$. Let ξ be real, but not algebraic of degree ≤ 2. Then there are infinitely many algebraic integers α of degree ≤ 3 which satisfy

$$0 < |\xi - \alpha| \ll H(\alpha)^{-\eta_3},$$

where

$$\eta_3 = \frac{1}{2}(3 + \sqrt{5}) = 2.618...$$

Conjecture. Let ξ be real, but is not algebraic of degree $\leq n$. Suppose $\epsilon > 0$. Then there are infinitely many real algebraic integers α of degree $\leq n$ with

$$|\xi - \alpha| \ll H(\alpha)^{-n+\epsilon}.$$

Theorem 14 (Roy, 2001). There exist real numbers ξ such that for any algebraic integer α of degree ≤ 3, we have

$$|\xi - \alpha| \gg H(\alpha)^{-(3/2)\eta_3}.$$

XI. Most Recent Result

THEOREM 15. For any real number $\xi \notin \mathbb{A}_3$ there exist infinitely many algebraic numbers $\alpha \in \mathbb{A}_3$ such that

$$|\xi - \alpha| \ll H(\alpha)^{-A},$$

where $A = 3.7475..$ is the largest root of the equation

$$2x^3 - 11x^2 + 11x + 8 = 0.$$