Augustin Cauchy (1789 - 1857)

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$$\boxed{\begin{aligned} a^3 + b^3 &\geq (a+b)ab \\ a^3 + c^3 &\geq (a+c)ac \\ b^3 + c^3 &\geq (b+c)bc \end{aligned}}$$

$$2a^3+2b^3+2c^3 \geq (a+b)ab+(a+c)ac+(b+c)bc$$

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$$a^2 + b^2 \geq 2ab$$

$$(a^2 + b^2)c \geq 2abc$$

$$(b^2 + c^2)a \geq 2abc$$

THEOREM: For any $a, b, c \geq 0$ we have

$$a^3 + b^3 + c^3 \geq 3abc$$

PROOF: We have

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$$\boxed{\begin{aligned} a^3 + b^3 &\geq (a+b)ab \\ a^3 + c^3 &\geq (a+c)ac \\ b^3 + c^3 &\geq (b+c)bc \end{aligned}}$$

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THEOREM: For any $a, b, c \geq 0$ we have

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PROOF: We have

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$$2a^3 + 2b^3 + 2c^3 \geq \boxed{a^2b + ab^2 + a^2c + ac^2 + b^2c + bc^2} \qquad \boxed{a^2c + b^2c + b^2a + c^2a + a^2b + c^2b} \geq 6abc$$

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$$2a^3 + 2b^3 + 2c^3 \geq 6abc$$

THEOREM: For any $a, b, c, d \geq 0$ we have

$$a^4 + b^4 + c^4 + d^4 \geq 4abcd$$

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PROOF: Since $x + y \geq 2\sqrt{xy}$

THEOREM: For any $a, b, c, d \geq 0$ we have

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PROOF: Since $x + y \geq 2\sqrt{xy}$**, we have**

$$a^4 + b^4 \geq 2a^2b^2$$

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PROOF: Since $x + y \geq 2\sqrt{xy}$, we have

$$a^4 + b^4 \geq 2a^2b^2$$

$$c^4 + d^4 \geq 2c^2d^2$$

THEOREM: For any $a, b, c, d \geq 0$ we have

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$$\begin{aligned} a^4 + b^4 + c^4 + d^4 &\geq 2a^2b^2 + 2c^2d^2 \\ &\geq 2\sqrt{4a^2b^2c^2d^2} \end{aligned}$$

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$$\begin{aligned} a^4 + b^4 + c^4 + d^4 &\geq 2a^2b^2 + 2c^2d^2 \\ &\geq 2\sqrt{4a^2b^2c^2d^2} \\ &= 4abcd \end{aligned}$$

THEOREM: For any $a_1, a_2, \ldots, a_8 \geq 0$ we have

$$a_1^8 + a_2^8 + \ldots + a_8^8 \geq 8a_1a_2\ldots a_8$$

THEOREM: For any $a_1, a_2, \ldots, a_8 \geq 0$ we have

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PROOF: Since $x + y \geq 2\sqrt{xy}$, we have

$$a_1^8 + a_2^8 \geq 2a_1^4a_2^4$$

THEOREM: **For any** $a_1, a_2, \ldots, a_8 \geq 0$ **we have**

$$a_1^8 + a_2^8 + \ldots + a_8^8 \geq 8a_1 a_2 \ldots a_8$$

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PROOF: Since $x + y \geq 2\sqrt{xy}$, we have

$$a_1^8 + a_2^8 \geq 2a_1^4 a_2^4$$

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$$a_5^8 + a_6^8 \geq 2a_5^4 a_6^4$$

THEOREM: **For any** $a_1, a_2, \ldots, a_8 \geq 0$ **we have**

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$$a_7^8 + a_8^8 \geq 2a_7^4a_8^4$$

THEOREM: For any $a_1, a_2, \dots, a_8 \geq 0$ we have

$$a_1^8 + a_2^8 + \dots + a_8^8 \geq 8 a_1 a_2 \dots a_8$$

PROOF: Since $x + y \geq 2\sqrt{xy}$, we have

$$\boxed{\begin{aligned} a_1^8 + a_2^8 &\geq 2a_1^4 a_2^4 \\ a_3^8 + a_4^8 &\geq 2a_3^4 a_4^4 \\ a_5^8 + a_6^8 &\geq 2a_5^4 a_6^4 \\ a_7^8 + a_8^8 &\geq 2a_7^4 a_8^4 \end{aligned}}$$

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$$a_1^8 + a_2^8 + \ldots + a_8^8 \quad \geq 2a_1^4a_2^4 + 2a_3^4a_4^4 + 2a_5^4a_6^4 + 2a_7^4a_8^4$$

THEOREM: For any $a_1, a_2, \ldots, a_8 \geq 0$ we have

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$$\begin{aligned} a_1^8 + a_2^8 + \ldots + a_8^8 &\geq 2a_1^4a_2^4 + 2a_3^4a_4^4 + 2a_5^4a_6^4 + 2a_7^4a_8^4 \\ &\geq 2\sqrt{4a_1^4a_2^4a_3^4a_4^4} + 2\sqrt{4a_5^4a_6^4a_7^4a_8^4} \end{aligned}$$

THEOREM: For any $a_1, a_2, \ldots, a_8 \geq 0$ we have

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$$\begin{aligned} a_1^8 + a_2^8 + \ldots + a_8^8 \quad &\geq 2a_1^4a_2^4 + 2a_3^4a_4^4 + 2a_5^4a_6^4 + 2a_7^4a_8^4 \\ &\geq 2\sqrt{4a_1^4a_2^4a_3^4a_4^4} + 2\sqrt{4a_5^4a_6^4a_7^4a_8^4} \\ &= 4a_1^2a_2^2a_3^2a_4^2 + 4a_5^2a_6^2a_7^2a_8^2 \end{aligned}$$

THEOREM: For any $a_1, a_2, \ldots, a_8 \geq 0$ we have

$$a_1^8 + a_2^8 + \ldots + a_8^8 \geq 8a_1 a_2 \ldots a_8$$

PROOF: Since $x + y \geq 2\sqrt{xy}$, we have

$$\boxed{\begin{aligned} a_1^8 + a_2^8 &\geq 2a_1^4 a_2^4 \\ a_3^8 + a_4^8 &\geq 2a_3^4 a_4^4 \\ a_5^8 + a_6^8 &\geq 2a_5^4 a_6^4 \\ a_7^8 + a_8^8 &\geq 2a_7^4 a_8^4 \end{aligned}}$$

$$\begin{aligned} a_1^8 + a_2^8 + \ldots + a_8^8 &\geq 2a_1^4 a_2^4 + 2a_3^4 a_4^4 + 2a_5^4 a_6^4 + 2a_7^4 a_8^4 \\ &\geq 2\sqrt{4a_1^4 a_2^4 a_3^4 a_4^4} + 2\sqrt{4a_5^4 a_6^4 a_7^4 a_8^4} \\ &= 4a_1^2 a_2^2 a_3^2 a_4^2 + 4a_5^2 a_6^2 a_7^2 a_8^2 \\ &\geq 2\sqrt{16a_1^2 a_2^2 \ldots a_8^2} \end{aligned}$$

THEOREM: For any $a_1, a_2, \ldots, a_8 \geq 0$ we have

$$a_1^8 + a_2^8 + \ldots + a_8^8 \geq 8a_1a_2\ldots a_8$$

PROOF: Since $x + y \geq 2\sqrt{xy}$, we have

$$\boxed{\begin{aligned} a_1^8 + a_2^8 &\geq 2a_1^4a_2^4 \\ a_3^8 + a_4^8 &\geq 2a_3^4a_4^4 \\ a_5^8 + a_6^8 &\geq 2a_5^4a_6^4 \\ a_7^8 + a_8^8 &\geq 2a_7^4a_8^4 \end{aligned}}$$

$$\begin{aligned} a_1^8 + a_2^8 + \ldots + a_8^8 &\geq 2a_1^4a_2^4 + 2a_3^4a_4^4 + 2a_5^4a_6^4 + 2a_7^4a_8^4 \\ &\geq 2\sqrt{4a_1^4a_2^4a_3^4a_4^4} + 2\sqrt{4a_5^4a_6^4a_7^4a_8^4} \\ &= 4a_1^2a_2^2a_3^2a_4^2 + 4a_5^2a_6^2a_7^2a_8^2 \\ &\geq 2\sqrt{16a_1^2a_2^2\ldots a_8^2} = 8a_1a_2\ldots a_8 \end{aligned}$$

THEOREM: If $n = 2^k$, then for any $a_1, a_2, \ldots, a_n \geq 0$ we have

$$a_1^n + a_2^n + \ldots + a_n^n \geq n a_1 a_2 \ldots a_n$$

THEOREM: If $n = 2^k$, then for any $a_1, a_2, \ldots, a_n \geq 0$ we have

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LEMMA:

THEOREM: If $n = 2^k$, then for any $a_1, a_2, \ldots, a_n \geq 0$ we have

$$a_1^n + a_2^n + \ldots + a_n^n \geq n a_1 a_2 \ldots a_n$$

LEMMA: Suppose

$$a_1^n + \ldots + a_n^n \geq n a_1 \ldots a_n$$

for any $a_1, a_2, \ldots, a_n \geq 0$.

THEOREM: If $n = 2^k$, then for any $a_1, a_2, \ldots, a_n \geq 0$ we have

$$a_1^n + a_2^n + \ldots + a_n^n \geq n a_1 a_2 \ldots a_n$$

LEMMA: Suppose

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for any $a_1, a_2, \ldots, a_n \geq 0$. Then

$$b_1^{n-1} + \ldots + b_{n-1}^{n-1} \geq (n-1) b_1 \ldots b_{n-1}$$

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Assume to the contrary that this function is < 0 **for some** $x_1, \ldots, x_n, y_1, \ldots, y_n \in [A, B]$. **Suppose that** f **attains its minimal value on** $[A, B]$ **at some** $a_1, \ldots, a_n, b_1, \ldots, b_n \in [A, B]$

THEOREM (Cauchy - Schwartz): For any $a_1, \ldots, a_n \geq 0$ and any $b_1, \ldots, b_n \geq 0$ we have

$$(a_1^2 + \ldots + a_n^2)(b_1^2 + \ldots + b_n^2) \geq (a_1 b_1 + \ldots + a_n b_n)^2$$

PROOF: Put

$$f(x_1, \ldots, x_n, y_1, \ldots, y_n) = (x_1^2 + \ldots + x_n^2)(y_1^2 + \ldots + y_n^2)$$
$$-(x_1 y_1 + \ldots + x_n y_n)^2$$

Assume to the contrary that this function is < 0 for some $x_1, \ldots, x_n, y_1, \ldots, y_n \in [A, B]$. Suppose that f attains its minimal value on $[A, B]$ at some $a_1, \ldots, a_n, b_1, \ldots, b_n \in [A, B]$ with $a_1 \neq a_2$

THEOREM (Cauchy - Schwartz): For any $a_1, \ldots, a_n \geq 0$ **and any** $b_1, \ldots, b_n \geq 0$ **we have**

$$(a_1^2 + \ldots + a_n^2)(b_1^2 + \ldots + b_n^2) \geq (a_1 b_1 + \ldots + a_n b_n)^2$$

PROOF: Put

$$f(x_1, \ldots, x_n, y_1, \ldots, y_n) = (x_1^2 + \ldots + x_n^2)(y_1^2 + \ldots + y_n^2) - (x_1 y_1 + \ldots + x_n y_n)^2$$

Assume to the contrary that this function is < 0 **for some** $x_1, \ldots, x_n, y_1, \ldots, y_n \in [A, B]$. **Suppose that** f **attains its minimal value on** $[A, B]$ **at some** $a_1, \ldots, a_n, b_1, \ldots, b_n \in [A, B]$ **with** $a_1 \neq a_2$ **or** $b_1 \neq b_2$.

THEOREM (Cauchy - Schwartz): For any $a_1, \ldots, a_n \geq 0$ **and any** $b_1, \ldots, b_n \geq 0$ **we have**

$$(a_1^2 + \ldots + a_n^2)(b_1^2 + \ldots + b_n^2) \geq (a_1 b_1 + \ldots + a_n b_n)^2$$

PROOF: Put

$$f(x_1, \ldots, x_n, y_1, \ldots, y_n) = (x_1^2 + \ldots + x_n^2)(y_1^2 + \ldots + y_n^2) - (x_1 y_1 + \ldots + x_n y_n)^2$$

Assume to the contrary that this function is < 0 **for some** $x_1, \ldots, x_n, y_1, \ldots, y_n \in [A, B]$. **Suppose that** f **attains its minimal value on** $[A, B]$ **at some** $a_1, \ldots, a_n, b_1, \ldots, b_n \in [A, B]$ **with** $a_1 \neq a_2$ **or** $b_1 \neq b_2$. **This gives us a contradiction, since**

$$f(a_1, a_2, a_3, \ldots, a_n, b_1, b_2, b_3, \ldots, b_n)$$
$$> f\left(\frac{a_1 + a_2}{2}, \frac{a_1 + a_2}{2}, a_3, \ldots, a_n, \frac{b_1 + b_2}{2}, \frac{b_1 + b_2}{2}, b_3, \ldots, b_n\right) \quad \blacksquare$$

THEOREM (Cauchy - Schwartz): For any $a_1, \ldots, a_n \geq 0$ and any $b_1, \ldots, b_n \geq 0$ we have

$$(a_1^2 + \ldots + a_n^2)(b_1^2 + \ldots + b_n^2) \geq (a_1 b_1 + \ldots + a_n b_n)^2$$

PROOF: Put

$$\begin{aligned} f(x_1, \ldots, x_n, y_1, \ldots, y_n) = & \ (x_1^2 + \ldots + x_n^2)(y_1^2 + \ldots + y_n^2) \\ & - (x_1 y_1 + \ldots + x_n y_n)^2 \end{aligned}$$

Assume to the contrary that this function is < 0 for some $x_1, \ldots, x_n, y_1, \ldots, y_n \in [A, B]$.

Suppose that f attains its minimal value on $[A, B]$ at some $a_1, \ldots, a_n, b_1, \ldots, b_n \in [A, B]$ with $a_1 \neq a_2$ or $b_1 \neq b_2$. This gives us a contradiction, since

$$\begin{aligned} & f(a_1, a_2, a_3, \ldots, a_n, b_1, b_2, b_3, \ldots, b_n) \\ & > f\left(\frac{a_1 + a_2}{2}, \frac{a_1 + a_2}{2}, a_3, \ldots, a_n, \frac{b_1 + b_2}{2}, \frac{b_1 + b_2}{2}, b_3, \ldots, b_n\right) \quad \blacksquare \end{aligned}$$

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PROOF: Put

$$f(x_1, \ldots, x_n, y_1, \ldots, y_n) = (x_1^2 + \ldots + x_n^2)(y_1^2 + \ldots + y_n^2) - (x_1 y_1 + \ldots + x_n y_n)^2$$

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Suppose that f attains its minimal value on $[A, B]$ at some $a_1, \ldots, a_n, b_1, \ldots, b_n \in [A, B]$ with $a_1 \neq a_2$ or $b_1 \neq b_2$. This gives us a contradiction, since

$$f(a_1, a_2, a_3, \ldots, a_n, b_1, b_2, b_3, \ldots, b_n)$$

$$> f\left(\frac{a_1 + a_2}{2}, \frac{a_1 + a_2}{2}, a_3, \ldots, a_n, \frac{b_1 + b_2}{2}, \frac{b_1 + b_2}{2}, b_3, \ldots, b_n\right) \quad \blacksquare$$

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PROOF: Put

$$f(x_1, \dots, x_n, y_1, \dots, y_n) = (x_1^2 + \dots + x_n^2)(y_1^2 + \dots + y_n^2) - (x_1 y_1 + \dots + x_n y_n)^2$$

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Suppose that f attains its minimal value on $[A, B]$ at some $a_1, \dots, a_n, b_1, \dots, b_n \in [A, B]$ with $a_1 \neq a_2$ or $b_1 \neq b_2$. This gives us a contradiction, since

$$f(a_1, a_2, a_3, \dots, a_n, b_1, b_2, b_3, \dots, b_n) > f\left(\frac{a_1 + a_2}{2}, \frac{a_1 + a_2}{2}, a_3, \dots, a_n, \frac{b_1 + b_2}{2}, \frac{b_1 + b_2}{2}, b_3, \dots, b_n\right) \quad \blacksquare$$

$$f(a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n)$$

$$= (a_1^2 + a_2^2 + \dots + a_n^2)(b_1^2 + b_2^2 + \dots + b_n^2) - (a_1 b_1 + a_2 b_2 + \dots + a_n b_n)^2$$

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Assume to the contrary that this function is < 0 **for some** $x_1, \ldots, x_n, y_1, \ldots, y_n \in [A, B]$.

Suppose that f **attains its minimal value on** $[A, B]$ **at some** $a_1, \ldots, a_n, b_1, \ldots, b_n \in [A, B]$

with $a_1 \neq a_2$ **or** $b_1 \neq b_2$. **This gives us a contradiction, since**

$$f(a_1, a_2, a_3, \ldots, a_n, b_1, b_2, b_3, \ldots, b_n)$$

$$> f\left(\frac{a_1 + a_2}{2}, \frac{a_1 + a_2}{2}, a_3, \ldots, a_n, \frac{b_1 + b_2}{2}, \frac{b_1 + b_2}{2}, b_3, \ldots, b_n\right) \quad \blacksquare$$

$$f(a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n)$$

$$= (a_1^2 + a_2^2 + \ldots + a_n^2)(b_1^2 + b_2^2 + \ldots + b_n^2) - (a_1 b_1 + a_2 b_2 + \ldots + a_n b_n)^2$$

$$f\left(\frac{a_1 + a_2}{2}, \frac{a_1 + a_2}{2}, a_3, \ldots, a_n, \frac{b_1 + b_2}{2}, \frac{b_1 + b_2}{2}, b_3, \ldots, b_n\right)$$

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PROOF: Put

$$f(x_1, \ldots, x_n, y_1, \ldots, y_n) = (x_1^2 + \ldots + x_n^2)(y_1^2 + \ldots + y_n^2) - (x_1 y_1 + \ldots + x_n y_n)^2$$

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$$f(a_1, a_2, a_3, \ldots, a_n, b_1, b_2, b_3, \ldots, b_n)$$

$$> f\left(\frac{a_1 + a_2}{2}, \frac{a_1 + a_2}{2}, a_3, \ldots, a_n, \frac{b_1 + b_2}{2}, \frac{b_1 + b_2}{2}, b_3, \ldots, b_n\right) \quad \blacksquare$$

$$f(a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n)$$

$$= (a_1^2 + a_2^2 + \ldots + a_n^2)(b_1^2 + b_2^2 + \ldots + b_n^2) - (a_1 b_1 + a_2 b_2 + \ldots + a_n b_n)^2$$

$$f\left(\frac{a_1 + a_2}{2}, \frac{a_1 + a_2}{2}, a_3, \ldots, a_n, \frac{b_1 + b_2}{2}, \frac{b_1 + b_2}{2}, b_3, \ldots, b_n\right)$$

$$= \left(\left(\frac{a_1 + a_2}{2}\right)^2 + \left(\frac{a_1 + a_2}{2}\right)^2 + \ldots + a_n^2\right)\left(\left(\frac{b_1 + b_2}{2}\right)^2 + \left(\frac{b_1 + b_2}{2}\right)^2 + \ldots + b_n^2\right)$$

THEOREM (Cauchy - Schwartz): For any $a_1, \dots, a_n \geq 0$ and any $b_1, \dots, b_n \geq 0$ we have

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PROOF: Put

$$\begin{aligned} f(x_1, \dots, x_n, y_1, \dots, y_n) = \; & (x_1^2 + \dots + x_n^2)(y_1^2 + \dots + y_n^2) \\ & -(x_1 y_1 + \dots + x_n y_n)^2 \end{aligned}$$

Assume to the contrary that this function is < 0 for some $x_1, \dots, x_n, y_1, \dots, y_n \in [A, B]$.

Suppose that f attains its minimal value on $[A, B]$ at some $a_1, \dots, a_n, b_1, \dots, b_n \in [A, B]$ with $a_1 \neq a_2$ or $b_1 \neq b_2$. This gives us a contradiction, since

$$\begin{aligned} & f(a_1, a_2, a_3, \dots, a_n, b_1, b_2, b_3, \dots, b_n) \\ & > f\left(\frac{a_1 + a_2}{2}, \frac{a_1 + a_2}{2}, a_3, \dots, a_n, \frac{b_1 + b_2}{2}, \frac{b_1 + b_2}{2}, b_3, \dots, b_n\right) \quad \blacksquare \end{aligned}$$

$$f(a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n)$$

$$= (a_1^2 + a_2^2 + \dots + a_n^2)(b_1^2 + b_2^2 + \dots + b_n^2) - (a_1 b_1 + a_2 b_2 + \dots + a_n b_n)^2$$

$$f\left(\frac{a_1 + a_2}{2}, \frac{a_1 + a_2}{2}, a_3, \dots, a_n, \frac{b_1 + b_2}{2}, \frac{b_1 + b_2}{2}, b_3, \dots, b_n\right)$$

$$= \left(\left(\frac{a_1 + a_2}{2}\right)^2 + \left(\frac{a_1 + a_2}{2}\right)^2 + \dots + a_n^2\right)\left(\left(\frac{b_1 + b_2}{2}\right)^2 + \left(\frac{b_1 + b_2}{2}\right)^2 + \dots + b_n^2\right)$$

$$- \left(\frac{a_1 + a_2}{2}\frac{b_1 + b_2}{2} + \frac{a_1 + a_2}{2}\frac{b_1 + b_2}{2} + \dots + a_n b_n\right)^2$$

with $a_1 \neq a_2$ or $b_1 \neq b_2$. This gives us a contradiction, since

$$f(a_1, a_2, a_3, \ldots, a_n, b_1, b_2, b_3, \ldots, b_n)$$

$$> f\left(\frac{a_1 + a_2}{2}, \frac{a_1 + a_2}{2}, a_3, \ldots, a_n, \frac{b_1 + b_2}{2}, \frac{b_1 + b_2}{2}, b_3, \ldots, b_n\right) \quad \blacksquare$$

$$f(a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n)$$

$$= (a_1^2 + a_2^2 + \ldots + a_n^2)(b_1^2 + b_2^2 + \ldots + b_n^2) - (a_1 b_1 + a_2 b_2 + \ldots + a_n b_n)^2$$

$$f\left(\frac{a_1 + a_2}{2}, \frac{a_1 + a_2}{2}, a_3, \ldots, a_n, \frac{b_1 + b_2}{2}, \frac{b_1 + b_2}{2}, b_3, \ldots, b_n\right)$$

$$= \left(\left(\frac{a_1 + a_2}{2}\right)^2 + \left(\frac{a_1 + a_2}{2}\right)^2 + \ldots + a_n^2\right)\left(\left(\frac{b_1 + b_2}{2}\right)^2 + \left(\frac{b_1 + b_2}{2}\right)^2 + \ldots + b_n^2\right)$$

$$- \left(\frac{a_1 + a_2}{2}\frac{b_1 + b_2}{2} + \frac{a_1 + a_2}{2}\frac{b_1 + b_2}{2} + \ldots + a_n b_n\right)^2$$

$$a_1^2 + a_2^2 > \left(\frac{a_1+a_2}{2}\right)^2 + \left(\frac{a_1+a_2}{2}\right)^2$$

with $a_1 \neq a_2$ or $b_1 \neq b_2$. This gives us a contradiction, since

$$f(a_1, a_2, a_3, \ldots, a_n, b_1, b_2, b_3, \ldots, b_n)$$
$$> f\left(\frac{a_1+a_2}{2}, \frac{a_1+a_2}{2}, a_3, \ldots, a_n, \frac{b_1+b_2}{2}, \frac{b_1+b_2}{2}, b_3, \ldots, b_n\right) \quad \blacksquare$$

$$f(a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n)$$
$$= (a_1^2 + a_2^2 + \ldots + a_n^2)(b_1^2 + b_2^2 + \ldots + b_n^2) - (a_1 b_1 + a_2 b_2 + \ldots + a_n b_n)^2$$

$$f\left(\frac{a_1+a_2}{2}, \frac{a_1+a_2}{2}, a_3, \ldots, a_n, \frac{b_1+b_2}{2}, \frac{b_1+b_2}{2}, b_3, \ldots, b_n\right)$$
$$= \left(\left(\frac{a_1+a_2}{2}\right)^2 + \left(\frac{a_1+a_2}{2}\right)^2 + \ldots + a_n^2\right)\left(\left(\frac{b_1+b_2}{2}\right)^2 + \left(\frac{b_1+b_2}{2}\right)^2 + \ldots + b_n^2\right)$$
$$- \left(\frac{a_1+a_2}{2}\frac{b_1+b_2}{2} + \frac{a_1+a_2}{2}\frac{b_1+b_2}{2} + \ldots + a_n b_n\right)^2$$

$$a_1^2 + a_2^2 > \left(\frac{a_1 + a_2}{2}\right)^2 + \left(\frac{a_1 + a_2}{2}\right)^2$$

$$a_1^2 + a_2^2 > 2\left(\frac{a_1 + a_2}{2}\right)^2$$

with $a_1 \neq a_2$ or $b_1 \neq b_2$. This gives us a contradiction, since

$$f(a_1, a_2, a_3, \ldots, a_n, b_1, b_2, b_3, \ldots, b_n)$$

$$> f\left(\frac{a_1 + a_2}{2}, \frac{a_1 + a_2}{2}, a_3, \ldots, a_n, \frac{b_1 + b_2}{2}, \frac{b_1 + b_2}{2}, b_3, \ldots, b_n\right) \quad \blacksquare$$

$$f(a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n)$$

$$= (a_1^2 + a_2^2 + \ldots + a_n^2)(b_1^2 + b_2^2 + \ldots + b_n^2) - (a_1 b_1 + a_2 b_2 + \ldots + a_n b_n)^2$$

$$f\left(\frac{a_1 + a_2}{2}, \frac{a_1 + a_2}{2}, a_3, \ldots, a_n, \frac{b_1 + b_2}{2}, \frac{b_1 + b_2}{2}, b_3, \ldots, b_n\right)$$

$$= \left(\left(\frac{a_1 + a_2}{2}\right)^2 + \left(\frac{a_1 + a_2}{2}\right)^2 + \ldots + a_n^2\right)\left(\left(\frac{b_1 + b_2}{2}\right)^2 + \left(\frac{b_1 + b_2}{2}\right)^2 + \ldots + b_n^2\right)$$

$$- \left(\frac{a_1 + a_2}{2}\frac{b_1 + b_2}{2} + \frac{a_1 + a_2}{2}\frac{b_1 + b_2}{2} + \ldots + a_n b_n\right)^2$$

$$a_1^2 + a_2^2 > \left(\frac{a_1 + a_2}{2}\right)^2 + \left(\frac{a_1 + a_2}{2}\right)^2$$

$$a_1^2 + a_2^2 > 2\left(\frac{a_1 + a_2}{2}\right)^2$$

$$a_1^2 + a_2^2 > 2\frac{(a_1 + a_2)^2}{4}$$

with $a_1 \neq a_2$ or $b_1 \neq b_2$. This gives us a contradiction, since

$$f(a_1, a_2, a_3, \ldots, a_n, b_1, b_2, b_3, \ldots, b_n)$$
$$> f\left(\frac{a_1 + a_2}{2}, \frac{a_1 + a_2}{2}, a_3, \ldots, a_n, \frac{b_1 + b_2}{2}, \frac{b_1 + b_2}{2}, b_3, \ldots, b_n\right) \quad \blacksquare$$

$$f(a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n)$$

$$= (a_1^2 + a_2^2 + \ldots + a_n^2)(b_1^2 + b_2^2 + \ldots + b_n^2) - (a_1 b_1 + a_2 b_2 + \ldots + a_n b_n)^2$$

$$f\left(\frac{a_1 + a_2}{2}, \frac{a_1 + a_2}{2}, a_3, \ldots, a_n, \frac{b_1 + b_2}{2}, \frac{b_1 + b_2}{2}, b_3, \ldots, b_n\right)$$

$$= \left(\left(\frac{a_1 + a_2}{2}\right)^2 + \left(\frac{a_1 + a_2}{2}\right)^2 + \ldots + a_n^2\right)\left(\left(\frac{b_1 + b_2}{2}\right)^2 + \left(\frac{b_1 + b_2}{2}\right)^2 + \ldots + b_n^2\right)$$
$$- \left(\frac{a_1 + a_2}{2}\frac{b_1 + b_2}{2} + \frac{a_1 + a_2}{2}\frac{b_1 + b_2}{2} + \ldots + a_n b_n\right)^2$$

$$a_1^2 + a_2^2 > \left(\frac{a_1 + a_2}{2}\right)^2 + \left(\frac{a_1 + a_2}{2}\right)^2 \quad a_1^2 + a_2^2 > \frac{(a_1 + a_2)^2}{2}$$

$$a_1^2 + a_2^2 > 2\left(\frac{a_1 + a_2}{2}\right)^2$$

$$a_1^2 + a_2^2 > 2\frac{(a_1 + a_2)^2}{4}$$

with $a_1 \neq a_2$ or $b_1 \neq b_2$. This gives us a contradiction, since

$$f(a_1, a_2, a_3, \ldots, a_n, b_1, b_2, b_3, \ldots, b_n)$$
$$> f\left(\frac{a_1 + a_2}{2}, \frac{a_1 + a_2}{2}, a_3, \ldots, a_n, \frac{b_1 + b_2}{2}, \frac{b_1 + b_2}{2}, b_3, \ldots, b_n\right) \quad \blacksquare$$

$$f(a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n)$$

$$= (a_1^2 + a_2^2 + \ldots + a_n^2)(b_1^2 + b_2^2 + \ldots + b_n^2) - (a_1 b_1 + a_2 b_2 + \ldots + a_n b_n)^2$$

$$f\left(\frac{a_1 + a_2}{2}, \frac{a_1 + a_2}{2}, a_3, \ldots, a_n, \frac{b_1 + b_2}{2}, \frac{b_1 + b_2}{2}, b_3, \ldots, b_n\right)$$

$$= \left(\left(\frac{a_1 + a_2}{2}\right)^2 + \left(\frac{a_1 + a_2}{2}\right)^2 + \ldots + a_n^2\right)\left(\left(\frac{b_1 + b_2}{2}\right)^2 + \left(\frac{b_1 + b_2}{2}\right)^2 + \ldots + b_n^2\right)$$

$$- \left(\frac{a_1 + a_2}{2}\frac{b_1 + b_2}{2} + \frac{a_1 + a_2}{2}\frac{b_1 + b_2}{2} + \ldots + a_n b_n\right)^2$$

$$a_1^2+a_2^2 > \left(\frac{a_1+a_2}{2}\right)^2+\left(\frac{a_1+a_2}{2}\right)^2 \qquad a_1^2+a_2^2 > \frac{(a_1+a_2)^2}{2}$$

$$a_1^2+a_2^2 > 2\left(\frac{a_1+a_2}{2}\right)^2 \qquad 2(a_1^2+a_2^2) > (a_1+a_2)^2$$

$$a_1^2+a_2^2 > 2\frac{(a_1+a_2)^2}{4}$$

with $a_1 \neq a_2$ or $b_1 \neq b_2$. This gives us a contradiction, since

$$f(a_1, a_2, a_3, \ldots, a_n, b_1, b_2, b_3, \ldots, b_n)$$
$$> f\left(\frac{a_1+a_2}{2}, \frac{a_1+a_2}{2}, a_3, \ldots, a_n, \frac{b_1+b_2}{2}, \frac{b_1+b_2}{2}, b_3, \ldots, b_n\right) \quad \blacksquare$$

$$f(a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n)$$

$$= (a_1^2+a_2^2+\ldots+a_n^2)(b_1^2+b_2^2+\ldots+b_n^2)-(a_1b_1+a_2b_2+\ldots+a_nb_n)^2$$

$$f\left(\frac{a_1+a_2}{2}, \frac{a_1+a_2}{2}, a_3, \ldots, a_n, \frac{b_1+b_2}{2}, \frac{b_1+b_2}{2}, b_3, \ldots, b_n\right)$$

$$= \left(\left(\frac{a_1+a_2}{2}\right)^2+\left(\frac{a_1+a_2}{2}\right)^2+\ldots+a_n^2\right)\left(\left(\frac{b_1+b_2}{2}\right)^2+\left(\frac{b_1+b_2}{2}\right)^2+\ldots+b_n^2\right)$$
$$-\left(\frac{a_1+a_2}{2}\frac{b_1+b_2}{2}+\frac{a_1+a_2}{2}\frac{b_1+b_2}{2}+\ldots+a_nb_n\right)^2$$

$$a_1^2 + a_2^2 > \left(\frac{a_1 + a_2}{2}\right)^2 + \left(\frac{a_1 + a_2}{2}\right)^2 \qquad a_1^2 + a_2^2 > \frac{(a_1 + a_2)^2}{2}$$

$$a_1^2 + a_2^2 > 2\left(\frac{a_1 + a_2}{2}\right)^2 \qquad 2(a_1^2 + a_2^2) > (a_1 + a_2)^2$$

$$a_1^2 + a_2^2 > 2\frac{(a_1 + a_2)^2}{4} \qquad 2a_1^2 + 2a_2^2 > a_1^2 + 2a_1a_2 + a_2^2$$

with $a_1 \neq a_2$ or $b_1 \neq b_2$. This gives us a contradiction, since

$$f(a_1, a_2, a_3, \ldots, a_n, b_1, b_2, b_3, \ldots, b_n)$$
$$> f\left(\frac{a_1 + a_2}{2}, \frac{a_1 + a_2}{2}, a_3, \ldots, a_n, \frac{b_1 + b_2}{2}, \frac{b_1 + b_2}{2}, b_3, \ldots, b_n\right) \quad \blacksquare$$

$$f(a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n)$$

$$= (a_1^2 + a_2^2 + \ldots + a_n^2)(b_1^2 + b_2^2 + \ldots + b_n^2) - (a_1b_1 + a_2b_2 + \ldots + a_nb_n)^2$$

$$f\left(\frac{a_1 + a_2}{2}, \frac{a_1 + a_2}{2}, a_3, \ldots, a_n, \frac{b_1 + b_2}{2}, \frac{b_1 + b_2}{2}, b_3, \ldots, b_n\right)$$

$$= \left(\left(\frac{a_1 + a_2}{2}\right)^2 + \left(\frac{a_1 + a_2}{2}\right)^2 + \ldots + a_n^2\right)\left(\left(\frac{b_1 + b_2}{2}\right)^2 + \left(\frac{b_1 + b_2}{2}\right)^2 + \ldots + b_n^2\right)$$

$$- \left(\frac{a_1 + a_2}{2}\frac{b_1 + b_2}{2} + \frac{a_1 + a_2}{2}\frac{b_1 + b_2}{2} + \ldots + a_nb_n\right)^2$$

$$a_1^2 + a_2^2 > \left(\frac{a_1 + a_2}{2}\right)^2 + \left(\frac{a_1 + a_2}{2}\right)^2 \quad a_1^2 + a_2^2 > \frac{(a_1 + a_2)^2}{2} \quad a_1^2 + a_2^2 > 2a_1 a_2$$

$$a_1^2 + a_2^2 > 2\left(\frac{a_1 + a_2}{2}\right)^2 \quad 2(a_1^2 + a_2^2) > (a_1 + a_2)^2$$

$$a_1^2 + a_2^2 > 2\frac{(a_1 + a_2)^2}{4} \quad 2a_1^2 + 2a_2^2 > a_1^2 + 2a_1 a_2 + a_2^2$$

with $a_1 \neq a_2$ or $b_1 \neq b_2$. This gives us a contradiction, since

$$f(a_1, a_2, a_3, \ldots, a_n, b_1, b_2, b_3, \ldots, b_n)$$

$$> f\left(\frac{a_1 + a_2}{2}, \frac{a_1 + a_2}{2}, a_3, \ldots, a_n, \frac{b_1 + b_2}{2}, \frac{b_1 + b_2}{2}, b_3, \ldots, b_n\right) \quad \blacksquare$$

$$f(a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n)$$

$$= (a_1^2 + a_2^2 + \ldots + a_n^2)(b_1^2 + b_2^2 + \ldots + b_n^2) - (a_1 b_1 + a_2 b_2 + \ldots + a_n b_n)^2$$

$$f\left(\frac{a_1 + a_2}{2}, \frac{a_1 + a_2}{2}, a_3, \ldots, a_n, \frac{b_1 + b_2}{2}, \frac{b_1 + b_2}{2}, b_3, \ldots, b_n\right)$$

$$= \left(\left(\frac{a_1 + a_2}{2}\right)^2 + \left(\frac{a_1 + a_2}{2}\right)^2 + \ldots + a_n^2\right)\left(\left(\frac{b_1 + b_2}{2}\right)^2 + \left(\frac{b_1 + b_2}{2}\right)^2 + \ldots + b_n^2\right)$$

$$- \left(\frac{a_1 + a_2}{2}\frac{b_1 + b_2}{2} + \frac{a_1 + a_2}{2}\frac{b_1 + b_2}{2} + \ldots + a_n b_n\right)^2$$

$$a_1^2 + a_2^2 > \left(\frac{a_1 + a_2}{2}\right)^2 + \left(\frac{a_1 + a_2}{2}\right)^2 \qquad a_1^2 + a_2^2 > \frac{(a_1 + a_2)^2}{2} \qquad a_1^2 + a_2^2 > 2a_1a_2$$

$$a_1^2 + a_2^2 > 2\left(\frac{a_1 + a_2}{2}\right)^2 \qquad 2(a_1^2 + a_2^2) > (a_1 + a_2)^2 \qquad a_1^2 - 2a_1a_2 + a_2^2 > 0$$

$$a_1^2 + a_2^2 > 2\frac{(a_1 + a_2)^2}{4} \qquad 2a_1^2 + 2a_2^2 > a_1^2 + 2a_1a_2 + a_2^2$$

with $a_1 \neq a_2$ or $b_1 \neq b_2$. This gives us a contradiction, since

$$f(a_1, a_2, a_3, \ldots, a_n, b_1, b_2, b_3, \ldots, b_n)$$

$$> f\left(\frac{a_1 + a_2}{2}, \frac{a_1 + a_2}{2}, a_3, \ldots, a_n, \frac{b_1 + b_2}{2}, \frac{b_1 + b_2}{2}, b_3, \ldots, b_n\right) \quad \blacksquare$$

$$f(a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n)$$

$$= (a_1^2 + a_2^2 + \ldots + a_n^2)(b_1^2 + b_2^2 + \ldots + b_n^2) - (a_1b_1 + a_2b_2 + \ldots + a_nb_n)^2$$

$$f\left(\frac{a_1 + a_2}{2}, \frac{a_1 + a_2}{2}, a_3, \ldots, a_n, \frac{b_1 + b_2}{2}, \frac{b_1 + b_2}{2}, b_3, \ldots, b_n\right)$$

$$= \left(\left(\frac{a_1 + a_2}{2}\right)^2 + \left(\frac{a_1 + a_2}{2}\right)^2 + \ldots + a_n^2\right)\left(\left(\frac{b_1 + b_2}{2}\right)^2 + \left(\frac{b_1 + b_2}{2}\right)^2 + \ldots + b_n^2\right)$$

$$- \left(\frac{a_1 + a_2}{2}\frac{b_1 + b_2}{2} + \frac{a_1 + a_2}{2}\frac{b_1 + b_2}{2} + \ldots + a_nb_n\right)^2$$

$$a_1^2 + a_2^2 > \left(\frac{a_1 + a_2}{2}\right)^2 + \left(\frac{a_1 + a_2}{2}\right)^2 \quad a_1^2 + a_2^2 > \frac{(a_1 + a_2)^2}{2} \quad a_1^2 + a_2^2 > 2a_1 a_2$$

$$a_1^2 + a_2^2 > 2\left(\frac{a_1 + a_2}{2}\right)^2 \quad 2(a_1^2 + a_2^2) > (a_1 + a_2)^2 \quad a_1^2 - 2a_1 a_2 + a_2^2 > 0$$

$$a_1^2 + a_2^2 > 2\frac{(a_1 + a_2)^2}{4} \quad 2a_1^2 + 2a_2^2 > a_1^2 + 2a_1 a_2 + a_2^2 \quad (a_1 - a_2)^2 > 0$$

with $a_1 \neq a_2$ or $b_1 \neq b_2$. This gives us a contradiction, since

$$f(a_1, a_2, a_3, \ldots, a_n, b_1, b_2, b_3, \ldots, b_n)$$

$$> f\left(\frac{a_1 + a_2}{2}, \frac{a_1 + a_2}{2}, a_3, \ldots, a_n, \frac{b_1 + b_2}{2}, \frac{b_1 + b_2}{2}, b_3, \ldots, b_n\right) \quad \blacksquare$$

$$f(a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n)$$

$$= (a_1^2 + a_2^2 + \ldots + a_n^2)(b_1^2 + b_2^2 + \ldots + b_n^2) - (a_1 b_1 + a_2 b_2 + \ldots + a_n b_n)^2$$

$$f\left(\frac{a_1 + a_2}{2}, \frac{a_1 + a_2}{2}, a_3, \ldots, a_n, \frac{b_1 + b_2}{2}, \frac{b_1 + b_2}{2}, b_3, \ldots, b_n\right)$$

$$= \left(\left(\frac{a_1 + a_2}{2}\right)^2 + \left(\frac{a_1 + a_2}{2}\right)^2 + \ldots + a_n^2\right)\left(\left(\frac{b_1 + b_2}{2}\right)^2 + \left(\frac{b_1 + b_2}{2}\right)^2 + \ldots + b_n^2\right)$$

$$- \left(\frac{a_1 + a_2}{2}\frac{b_1 + b_2}{2} + \frac{a_1 + a_2}{2}\frac{b_1 + b_2}{2} + \ldots + a_n b_n\right)^2$$

THEOREM (Cauchy - Schwartz): For any $a_1, \ldots, a_n \geq 0$ and any $b_1, \ldots, b_n \geq 0$ we have

$$(a_1^2 + \ldots + a_n^2)(b_1^2 + \ldots + b_n^2) \geq (a_1 b_1 + \ldots + a_n b_n)^2$$

PROOF: Put

$$f(x_1, \ldots, x_n, y_1, \ldots, y_n) = (x_1^2 + \ldots + x_n^2)(y_1^2 + \ldots + y_n^2) - (x_1 y_1 + \ldots + x_n y_n)^2$$

Assume to the contrary that this function is < 0 for some $x_1, \ldots, x_n, y_1, \ldots, y_n \in [A, B]$.

Suppose that f attains its minimal value on $[A, B]$ at some $a_1, \ldots, a_n, b_1, \ldots, b_n \in [A, B]$ with $a_1 \neq a_2$ or $b_1 \neq b_2$. This gives us a contradiction, since

$$f(a_1, a_2, a_3, \ldots, a_n, b_1, b_2, b_3, \ldots, b_n)$$

$$> f\left(\frac{a_1 + a_2}{2}, \frac{a_1 + a_2}{2}, a_3, \ldots, a_n, \frac{b_1 + b_2}{2}, \frac{b_1 + b_2}{2}, b_3, \ldots, b_n\right) \quad \blacksquare$$

$$f(a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n)$$

$$= (a_1^2 + a_2^2 + \ldots + a_n^2)(b_1^2 + b_2^2 + \ldots + b_n^2) - (a_1 b_1 + a_2 b_2 + \ldots + a_n b_n)^2$$

$$f\left(\frac{a_1 + a_2}{2}, \frac{a_1 + a_2}{2}, a_3, \ldots, a_n, \frac{b_1 + b_2}{2}, \frac{b_1 + b_2}{2}, b_3, \ldots, b_n\right)$$

$$= \left(\left(\frac{a_1 + a_2}{2}\right)^2 + \left(\frac{a_1 + a_2}{2}\right)^2 + \ldots + a_n^2\right)\left(\left(\frac{b_1 + b_2}{2}\right)^2 + \left(\frac{b_1 + b_2}{2}\right)^2 + \ldots + b_n^2\right)$$

$$- \left(\frac{a_1 + a_2}{2}\frac{b_1 + b_2}{2} + \frac{a_1 + a_2}{2}\frac{b_1 + b_2}{2} + \ldots + a_n b_n\right)^2$$

$$f(a_1, a_2, a_3, \ldots, a_n, b_1, b_2, b_3, \ldots, b_n)$$

$$> f\left(\frac{a_1+a_2}{2}, \frac{a_1+a_2}{2}, a_3, \ldots, a_n, \frac{b_1+b_2}{2}, \frac{b_1+b_2}{2}, b_3, \ldots, b_n\right) \quad \blacksquare$$

$$f(a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n)$$

$$= (a_1^2 + a_2^2 + \ldots + a_n^2)(b_1^2 + b_2^2 + \ldots + b_n^2) - (a_1 b_1 + a_2 b_2 + \ldots + a_n b_n)^2$$

$$f\left(\frac{a_1+a_2}{2}, \frac{a_1+a_2}{2}, a_3, \ldots, a_n, \frac{b_1+b_2}{2}, \frac{b_1+b_2}{2}, b_3, \ldots, b_n\right)$$

$$= \left(\left(\frac{a_1+a_2}{2}\right)^2 + \left(\frac{a_1+a_2}{2}\right)^2 + \ldots + a_n^2\right)\left(\left(\frac{b_1+b_2}{2}\right)^2 + \left(\frac{b_1+b_2}{2}\right)^2 + \ldots + b_n^2\right)$$

$$- \left(\frac{a_1+a_2}{2}\frac{b_1+b_2}{2} + \frac{a_1+a_2}{2}\frac{b_1+b_2}{2} + \ldots + a_n b_n\right)^2$$

$$a_1 b_1 + a_2 b_2 < 2\frac{a_1 + a_2}{2}\frac{b_1 + b_2}{2}$$

with $a_1 \neq a_2$ or $b_1 \neq b_2$. This gives us a contradiction, since

$$f(a_1, a_2, a_3, \ldots, a_n, b_1, b_2, b_3, \ldots, b_n)$$

$$> f\left(\frac{a_1 + a_2}{2}, \frac{a_1 + a_2}{2}, a_3, \ldots, a_n, \frac{b_1 + b_2}{2}, \frac{b_1 + b_2}{2}, b_3, \ldots, b_n\right) \quad \blacksquare$$

$$f(a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n)$$

$$= (a_1^2 + a_2^2 + \ldots + a_n^2)(b_1^2 + b_2^2 + \ldots + b_n^2) - (a_1 b_1 + a_2 b_2 + \ldots + a_n b_n)^2$$

$$f\left(\frac{a_1 + a_2}{2}, \frac{a_1 + a_2}{2}, a_3, \ldots, a_n, \frac{b_1 + b_2}{2}, \frac{b_1 + b_2}{2}, b_3, \ldots, b_n\right)$$

$$= \left(\left(\frac{a_1 + a_2}{2}\right)^2 + \left(\frac{a_1 + a_2}{2}\right)^2 + \ldots + a_n^2\right)\left(\left(\frac{b_1 + b_2}{2}\right)^2 + \left(\frac{b_1 + b_2}{2}\right)^2 + \ldots + b_n^2\right)$$

$$- \left(\frac{a_1 + a_2}{2}\frac{b_1 + b_2}{2} + \frac{a_1 + a_2}{2}\frac{b_1 + b_2}{2} + \ldots + a_n b_n\right)^2$$

$$a_1 b_1 + a_2 b_2 < 2\frac{a_1 + a_2}{2}\frac{b_1 + b_2}{2}$$

$$a_1 b_1 + a_2 b_2 < \frac{(a_1 + a_2)(b_1 + b_2)}{2}$$

with $a_1 \neq a_2$ or $b_1 \neq b_2$. This gives us a contradiction, since

$$f(a_1, a_2, a_3, \ldots, a_n, b_1, b_2, b_3, \ldots, b_n)$$

$$> f\left(\frac{a_1 + a_2}{2}, \frac{a_1 + a_2}{2}, a_3, \ldots, a_n, \frac{b_1 + b_2}{2}, \frac{b_1 + b_2}{2}, b_3, \ldots, b_n\right) \quad \blacksquare$$

$$f(a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n)$$

$$= (a_1^2 + a_2^2 + \ldots + a_n^2)(b_1^2 + b_2^2 + \ldots + b_n^2) - (a_1 b_1 + a_2 b_2 + \ldots + a_n b_n)^2$$

$$f\left(\frac{a_1 + a_2}{2}, \frac{a_1 + a_2}{2}, a_3, \ldots, a_n, \frac{b_1 + b_2}{2}, \frac{b_1 + b_2}{2}, b_3, \ldots, b_n\right)$$

$$= \left(\left(\frac{a_1 + a_2}{2}\right)^2 + \left(\frac{a_1 + a_2}{2}\right)^2 + \ldots + a_n^2\right)\left(\left(\frac{b_1 + b_2}{2}\right)^2 + \left(\frac{b_1 + b_2}{2}\right)^2 + \ldots + b_n^2\right)$$

$$- \left(\frac{a_1 + a_2}{2}\frac{b_1 + b_2}{2} + \frac{a_1 + a_2}{2}\frac{b_1 + b_2}{2} + \ldots + a_n b_n\right)^2$$

$$a_1 b_1 + a_2 b_2 < 2\frac{a_1 + a_2}{2}\frac{b_1 + b_2}{2}$$

$$a_1 b_1 + a_2 b_2 < \frac{(a_1 + a_2)(b_1 + b_2)}{2}$$

$$2a_1 b_1 + 2a_2 b_2 < (a_1 + a_2)(b_1 + b_2)$$

with $a_1 \neq a_2$ or $b_1 \neq b_2$. This gives us a contradiction, since

$$f(a_1, a_2, a_3, \ldots, a_n, b_1, b_2, b_3, \ldots, b_n)$$

$$> f\left(\frac{a_1 + a_2}{2}, \frac{a_1 + a_2}{2}, a_3, \ldots, a_n, \frac{b_1 + b_2}{2}, \frac{b_1 + b_2}{2}, b_3, \ldots, b_n\right) \quad \blacksquare$$

$$f(a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n)$$

$$= (a_1^2 + a_2^2 + \ldots + a_n^2)(b_1^2 + b_2^2 + \ldots + b_n^2) - (a_1 b_1 + a_2 b_2 + \ldots + a_n b_n)^2$$

$$f\left(\frac{a_1 + a_2}{2}, \frac{a_1 + a_2}{2}, a_3, \ldots, a_n, \frac{b_1 + b_2}{2}, \frac{b_1 + b_2}{2}, b_3, \ldots, b_n\right)$$

$$= \left(\left(\frac{a_1 + a_2}{2}\right)^2 + \left(\frac{a_1 + a_2}{2}\right)^2 + \ldots + a_n^2\right)\left(\left(\frac{b_1 + b_2}{2}\right)^2 + \left(\frac{b_1 + b_2}{2}\right)^2 + \ldots + b_n^2\right)$$

$$- \left(\frac{a_1 + a_2}{2}\frac{b_1 + b_2}{2} + \frac{a_1 + a_2}{2}\frac{b_1 + b_2}{2} + \ldots + a_n b_n\right)^2$$

$$a_1b_1 + a_2b_2 < 2\frac{a_1 + a_2}{2}\frac{b_1 + b_2}{2}$$

$$a_1b_1 + a_2b_2 < \frac{(a_1 + a_2)(b_1 + b_2)}{2}$$

$$2a_1b_1 + 2a_2b_2 < (a_1 + a_2)(b_1 + b_2)$$

$$2a_1b_1 + 2a_2b_2 < a_1b_1 + a_1b_2 + a_2b_1 + a_2b_2$$

with $a_1 \neq a_2$ or $b_1 \neq b_2$. This gives us a contradiction, since

$$f(a_1, a_2, a_3, \ldots, a_n, b_1, b_2, b_3, \ldots, b_n)$$

$$> f\left(\frac{a_1 + a_2}{2}, \frac{a_1 + a_2}{2}, a_3, \ldots, a_n, \frac{b_1 + b_2}{2}, \frac{b_1 + b_2}{2}, b_3, \ldots, b_n\right) \quad \blacksquare$$

$$f(a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n)$$

$$= (a_1^2 + a_2^2 + \ldots + a_n^2)(b_1^2 + b_2^2 + \ldots + b_n^2) - (a_1b_1 + a_2b_2 + \ldots + a_nb_n)^2$$

$$f\left(\frac{a_1 + a_2}{2}, \frac{a_1 + a_2}{2}, a_3, \ldots, a_n, \frac{b_1 + b_2}{2}, \frac{b_1 + b_2}{2}, b_3, \ldots, b_n\right)$$

$$= \left(\left(\frac{a_1 + a_2}{2}\right)^2 + \left(\frac{a_1 + a_2}{2}\right)^2 + \ldots + a_n^2\right)\left(\left(\frac{b_1 + b_2}{2}\right)^2 + \left(\frac{b_1 + b_2}{2}\right)^2 + \ldots + b_n^2\right)$$

$$- \left(\frac{a_1 + a_2}{2}\frac{b_1 + b_2}{2} + \frac{a_1 + a_2}{2}\frac{b_1 + b_2}{2} + \ldots + a_nb_n\right)^2$$

$$a_1b_1 + a_2b_2 < 2\frac{a_1 + a_2}{2}\frac{b_1 + b_2}{2} \qquad a_1b_1 + a_2b_2 < a_1b_2 + a_2b_1$$

$$a_1b_1 + a_2b_2 < \frac{(a_1 + a_2)(b_1 + b_2)}{2}$$

$$2a_1b_1 + 2a_2b_2 < (a_1 + a_2)(b_1 + b_2)$$

$$2a_1b_1 + 2a_2b_2 < a_1b_1 + a_1b_2 + a_2b_1 + a_2b_2$$

with $a_1 \neq a_2$ or $b_1 \neq b_2$. This gives us a contradiction, since

$$f(a_1, a_2, a_3, \ldots, a_n, b_1, b_2, b_3, \ldots, b_n)$$
$$> f\left(\frac{a_1 + a_2}{2}, \frac{a_1 + a_2}{2}, a_3, \ldots, a_n, \frac{b_1 + b_2}{2}, \frac{b_1 + b_2}{2}, b_3, \ldots, b_n\right) \quad \blacksquare$$

$$f(a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n)$$

$$= (a_1^2 + a_2^2 + \ldots + a_n^2)(b_1^2 + b_2^2 + \ldots + b_n^2) - (a_1b_1 + a_2b_2 + \ldots + a_nb_n)^2$$

$$f\left(\frac{a_1 + a_2}{2}, \frac{a_1 + a_2}{2}, a_3, \ldots, a_n, \frac{b_1 + b_2}{2}, \frac{b_1 + b_2}{2}, b_3, \ldots, b_n\right)$$

$$= \left(\left(\frac{a_1 + a_2}{2}\right)^2 + \left(\frac{a_1 + a_2}{2}\right)^2 + \ldots + a_n^2\right)\left(\left(\frac{b_1 + b_2}{2}\right)^2 + \left(\frac{b_1 + b_2}{2}\right)^2 + \ldots + b_n^2\right)$$
$$- \left(\frac{a_1 + a_2}{2}\frac{b_1 + b_2}{2} + \frac{a_1 + a_2}{2}\frac{b_1 + b_2}{2} + \ldots + a_nb_n\right)^2$$

$$a_1 b_1 + a_2 b_2 < 2\frac{a_1 + a_2}{2}\frac{b_1 + b_2}{2}$$

$$a_1 b_1 + a_2 b_2 < \frac{(a_1 + a_2)(b_1 + b_2)}{2}$$

$$2a_1 b_1 + 2a_2 b_2 < (a_1 + a_2)(b_1 + b_2)$$

$$2a_1 b_1 + 2a_2 b_2 < a_1 b_1 + a_1 b_2 + a_2 b_1 + a_2 b_2$$

$$a_1 b_1 + a_2 b_2 < a_1 b_2 + a_2 b_1$$

$$a_1 b_1 - a_1 b_2 + a_2 b_2 - a_2 b_1 < 0$$

with $a_1 \neq a_2$ or $b_1 \neq b_2$. This gives us a contradiction, since

$$f(a_1, a_2, a_3, \ldots, a_n, b_1, b_2, b_3, \ldots, b_n)$$

$$> f\left(\frac{a_1 + a_2}{2}, \frac{a_1 + a_2}{2}, a_3, \ldots, a_n, \frac{b_1 + b_2}{2}, \frac{b_1 + b_2}{2}, b_3, \ldots, b_n\right) \quad \blacksquare$$

$$f(a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n)$$

$$= (a_1^2 + a_2^2 + \ldots + a_n^2)(b_1^2 + b_2^2 + \ldots + b_n^2) - (a_1 b_1 + a_2 b_2 + \ldots + a_n b_n)^2$$

$$f\left(\frac{a_1 + a_2}{2}, \frac{a_1 + a_2}{2}, a_3, \ldots, a_n, \frac{b_1 + b_2}{2}, \frac{b_1 + b_2}{2}, b_3, \ldots, b_n\right)$$

$$= \left(\left(\frac{a_1 + a_2}{2}\right)^2 + \left(\frac{a_1 + a_2}{2}\right)^2 + \ldots + a_n^2\right)\left(\left(\frac{b_1 + b_2}{2}\right)^2 + \left(\frac{b_1 + b_2}{2}\right)^2 + \ldots + b_n^2\right)$$

$$- \left(\frac{a_1 + a_2}{2}\frac{b_1 + b_2}{2} + \frac{a_1 + a_2}{2}\frac{b_1 + b_2}{2} + \ldots + a_n b_n\right)^2$$

$$a_1b_1 + a_2b_2 < 2\frac{a_1+a_2}{2}\frac{b_1+b_2}{2}$$

$$a_1b_1 + a_2b_2 < \frac{(a_1+a_2)(b_1+b_2)}{2}$$

$$2a_1b_1 + 2a_2b_2 < (a_1+a_2)(b_1+b_2)$$

$$2a_1b_1 + 2a_2b_2 < a_1b_1 + a_1b_2 + a_2b_1 + a_2b_2$$

$$a_1b_1 + a_2b_2 < a_1b_2 + a_2b_1$$

$$a_1b_1 - a_1b_2 + a_2b_2 - a_2b_1 < 0$$

$$a_1(b_1-b_2) + a_2(b_2-b_1) < 0$$

with $a_1 \neq a_2$ or $b_1 \neq b_2$. This gives us a contradiction, since

$$f(a_1, a_2, a_3, \ldots, a_n, b_1, b_2, b_3, \ldots, b_n)$$

$$> f\left(\frac{a_1+a_2}{2}, \frac{a_1+a_2}{2}, a_3, \ldots, a_n, \frac{b_1+b_2}{2}, \frac{b_1+b_2}{2}, b_3, \ldots, b_n\right) \quad \blacksquare$$

$$f(a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n)$$

$$= (a_1^2 + a_2^2 + \ldots + a_n^2)(b_1^2 + b_2^2 + \ldots + b_n^2) - (a_1b_1 + a_2b_2 + \ldots + a_nb_n)^2$$

$$f\left(\frac{a_1+a_2}{2}, \frac{a_1+a_2}{2}, a_3, \ldots, a_n, \frac{b_1+b_2}{2}, \frac{b_1+b_2}{2}, b_3, \ldots, b_n\right)$$

$$= \left(\left(\frac{a_1+a_2}{2}\right)^2 + \left(\frac{a_1+a_2}{2}\right)^2 + \ldots + a_n^2\right)\left(\left(\frac{b_1+b_2}{2}\right)^2 + \left(\frac{b_1+b_2}{2}\right)^2 + \ldots + b_n^2\right)$$

$$-\left(\frac{a_1+a_2}{2}\frac{b_1+b_2}{2} + \frac{a_1+a_2}{2}\frac{b_1+b_2}{2} + \ldots + a_nb_n\right)^2$$

$$a_1 b_1 + a_2 b_2 < 2\frac{a_1+a_2}{2}\frac{b_1+b_2}{2}$$

$$a_1 b_1 + a_2 b_2 < \frac{(a_1+a_2)(b_1+b_2)}{2}$$

$$2a_1 b_1 + 2a_2 b_2 < (a_1+a_2)(b_1+b_2)$$

$$2a_1 b_1 + 2a_2 b_2 < a_1 b_1 + a_1 b_2 + a_2 b_1 + a_2 b_2$$

$$a_1 b_1 + a_2 b_2 < a_1 b_2 + a_2 b_1$$

$$a_1 b_1 - a_1 b_2 + a_2 b_2 - a_2 b_1 < 0$$

$$a_1(b_1 - b_2) + a_2(b_2 - b_1) < 0$$

$$(a_1 - a_2)(b_1 - b_2) < 0$$

with $a_1 \neq a_2$ or $b_1 \neq b_2$. This gives us a contradiction, since

$$f(a_1, a_2, a_3, \ldots, a_n, b_1, b_2, b_3, \ldots, b_n)$$

$$> f\left(\frac{a_1+a_2}{2}, \frac{a_1+a_2}{2}, a_3, \ldots, a_n, \frac{b_1+b_2}{2}, \frac{b_1+b_2}{2}, b_3, \ldots, b_n\right) \quad \blacksquare$$

$$f(a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n)$$

$$= (a_1^2 + a_2^2 + \ldots + a_n^2)(b_1^2 + b_2^2 + \ldots + b_n^2) - (a_1 b_1 + a_2 b_2 + \ldots + a_n b_n)^2$$

$$f\left(\frac{a_1+a_2}{2}, \frac{a_1+a_2}{2}, a_3, \ldots, a_n, \frac{b_1+b_2}{2}, \frac{b_1+b_2}{2}, b_3, \ldots, b_n\right)$$

$$= \left(\left(\frac{a_1+a_2}{2}\right)^2 + \left(\frac{a_1+a_2}{2}\right)^2 + \ldots + a_n^2\right)\left(\left(\frac{b_1+b_2}{2}\right)^2 + \left(\frac{b_1+b_2}{2}\right)^2 + \ldots + b_n^2\right)$$

$$-\left(\frac{a_1+a_2}{2}\frac{b_1+b_2}{2} + \frac{a_1+a_2}{2}\frac{b_1+b_2}{2} + \ldots + a_n b_n\right)^2$$

THEOREM (Cauchy - Schwartz): For any $a_1, \ldots, a_n \geq 0$ **and any** $b_1, \ldots, b_n \geq 0$ **we have**

$$(a_1^2 + \ldots + a_n^2)(b_1^2 + \ldots + b_n^2) \geq (a_1 b_1 + \ldots + a_n b_n)^2$$

PROOF: Put

$$f(x_1, \ldots, x_n, y_1, \ldots, y_n) = (x_1^2 + \ldots + x_n^2)(y_1^2 + \ldots + y_n^2) - (x_1 y_1 + \ldots + x_n y_n)^2$$

Assume to the contrary that this function is < 0 **for some** $x_1, \ldots, x_n, y_1, \ldots, y_n \in [A, B]$.

Suppose that f **attains its minimal value on** $[A, B]$ **at some** $a_1, \ldots, a_n, b_1, \ldots, b_n \in [A, B]$ **with** $a_1 \neq a_2$ **or** $b_1 \neq b_2$. **This gives us a contradiction, since**

$$f(a_1, a_2, a_3, \ldots, a_n, b_1, b_2, b_3, \ldots, b_n)$$

$$> f\left(\frac{a_1 + a_2}{2}, \frac{a_1 + a_2}{2}, a_3, \ldots, a_n, \frac{b_1 + b_2}{2}, \frac{b_1 + b_2}{2}, b_3, \ldots, b_n\right) \quad \blacksquare$$

$$f(a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n)$$

$$= (a_1^2 + a_2^2 + \ldots + a_n^2)(b_1^2 + b_2^2 + \ldots + b_n^2) - (a_1 b_1 + a_2 b_2 + \ldots + a_n b_n)^2$$

$$f\left(\frac{a_1 + a_2}{2}, \frac{a_1 + a_2}{2}, a_3, \ldots, a_n, \frac{b_1 + b_2}{2}, \frac{b_1 + b_2}{2}, b_3, \ldots, b_n\right)$$

$$= \left(\left(\frac{a_1 + a_2}{2}\right)^2 + \left(\frac{a_1 + a_2}{2}\right)^2 + \ldots + a_n^2\right)\left(\left(\frac{b_1 + b_2}{2}\right)^2 + \left(\frac{b_1 + b_2}{2}\right)^2 + \ldots + b_n^2\right)$$

$$- \left(\frac{a_1 + a_2}{2}\frac{b_1 + b_2}{2} + \frac{a_1 + a_2}{2}\frac{b_1 + b_2}{2} + \ldots + a_n b_n\right)^2$$

THEOREM (Cauchy - Schwartz): For any $a_1, \ldots, a_n \geq 0$ and any $b_1, \ldots, b_n \geq 0$ we have

$$(a_1^2 + \ldots + a_n^2)(b_1^2 + \ldots + b_n^2) \geq (a_1 b_1 + \ldots + a_n b_n)^2$$

PROOF: Put

$$\begin{aligned} f(x_1, \ldots, x_n, y_1, \ldots, y_n) = \; & (x_1^2 + \ldots + x_n^2)(y_1^2 + \ldots + y_n^2) \\ & -(x_1 y_1 + \ldots + x_n y_n)^2 \end{aligned}$$

Assume to the contrary that this function is < 0 for some $x_1, \ldots, x_n, y_1, \ldots, y_n \in [A, B]$.

Suppose that f attains its minimal value on $[A, B]$ at some $a_1, \ldots, a_n, b_1, \ldots, b_n \in [A, B]$ with $a_1 \neq a_2$ or $b_1 \neq b_2$. This gives us a contradiction, since

$$\xcancel{\begin{aligned} & f(a_1, a_2, a_3, \ldots, a_n, b_1, b_2, b_3, \ldots, b_n) \\ & > f\left(\frac{a_1 + a_2}{2}, \frac{a_1 + a_2}{2}, a_3, \ldots, a_n, \frac{b_1 + b_2}{2}, \frac{b_1 + b_2}{2}, b_3, \ldots, b_n\right) \end{aligned}} \quad \blacksquare$$

$$f(a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n)$$

$$= (a_1^2 + a_2^2 + \ldots + a_n^2)(b_1^2 + b_2^2 + \ldots + b_n^2) - (a_1 b_1 + a_2 b_2 + \ldots + a_n b_n)^2$$

$$f\left(\frac{a_1 + a_2}{2}, \frac{a_1 + a_2}{2}, a_3, \ldots, a_n, \frac{b_1 + b_2}{2}, \frac{b_1 + b_2}{2}, b_3, \ldots, b_n\right)$$

$$= \left(\left(\frac{a_1 + a_2}{2}\right)^2 + \left(\frac{a_1 + a_2}{2}\right)^2 + \ldots + a_n^2\right)\left(\left(\frac{b_1 + b_2}{2}\right)^2 + \left(\frac{b_1 + b_2}{2}\right)^2 + \ldots + b_n^2\right)$$

$$- \left(\frac{a_1 + a_2}{2} \frac{b_1 + b_2}{2} + \frac{a_1 + a_2}{2} \frac{b_1 + b_2}{2} + \ldots + a_n b_n\right)^2$$

THEOREM (Cauchy - Schwartz): For any $a_1, \ldots, a_n \geq 0$ and any $b_1, \ldots, b_n \geq 0$ we have

$$(a_1^2 + \ldots + a_n^2)(b_1^2 + \ldots + b_n^2) \geq (a_1 b_1 + \ldots + a_n b_n)^2$$

PROOF: Put

$$f(x_1, \ldots, x_n, y_1, \ldots, y_n) = (x_1^2 + \ldots + x_n^2)(y_1^2 + \ldots + y_n^2) - (x_1 y_1 + \ldots + x_n y_n)^2$$

Assume to the contrary that this function is <0 for some $x_1, \ldots, x_n, y_1, \ldots, y_n \in [A, B]$.

Suppose that f attains its minimal value on $[A, B]$ at some $a_1, \ldots, a_n, b_1, \ldots, b_n \in [A, B]$ with $a_1 \neq a_2$ or $b_1 \neq b_2$. This gives us a contradiction, since

$$f(a_1, a_2, a_3, \ldots, a_n, b_1, b_2, b_3, \ldots, b_n)$$

$$> f\left(\sqrt{a_1 b_1}, \sqrt{a_2 b_2}, a_3, \ldots, a_n, \sqrt{a_1 b_1}, \sqrt{a_2 b_2}, b_3, \ldots, b_n\right) \quad \blacksquare$$

$$f(a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n)$$

$$= (a_1^2 + a_2^2 + \ldots + a_n^2)(b_1^2 + b_2^2 + \ldots + b_n^2) - (a_1 b_1 + a_2 b_2 + \ldots + a_n b_n)^2$$

$$f\left(\sqrt{a_1 b_1}, \sqrt{a_2 b_2}, a_3, \ldots, a_n, \sqrt{a_1 b_1}, \sqrt{a_2 b_2}, b_3, \ldots, b_n\right)$$

$$= \left(\left(\sqrt{a_1 b_1}\right)^2 + \left(\sqrt{a_2 b_2}\right)^2 + \ldots\right)\left(\left(\sqrt{a_1 b_1}\right)^2 + \left(\sqrt{a_2 b_2}\right)^2 + \ldots\right)$$

$$- \left(\sqrt{a_1 b_1}\sqrt{a_1 b_1} + \sqrt{a_2 b_2}\sqrt{a_2 b_2} + \ldots\right)^2$$

THEOREM (Cauchy - Schwartz): For any $a_1, \ldots, a_n \geq 0$ and any $b_1, \ldots, b_n \geq 0$ we have

$$(a_1^2 + \ldots + a_n^2)(b_1^2 + \ldots + b_n^2) \geq (a_1 b_1 + \ldots + a_n b_n)^2$$

PROOF: Put

$$f(x_1, \ldots, x_n, y_1, \ldots, y_n) = (x_1^2 + \ldots + x_n^2)(y_1^2 + \ldots + y_n^2) - (x_1 y_1 + \ldots + x_n y_n)^2$$

Assume to the contrary that this function is < 0 for some $x_1, \ldots, x_n, y_1, \ldots, y_n \in [A, B]$.

Suppose that f attains its minimal value on $[A, B]$ at some $a_1, \ldots, a_n, b_1, \ldots, b_n \in [A, B]$ with $a_1 \neq a_2$ or $b_1 \neq b_2$. This gives us a contradiction, since

$$f(a_1, a_2, a_3, \ldots, a_n, b_1, b_2, b_3, \ldots, b_n)$$

$$> f\left(\sqrt{\frac{a_1^2 + a_2^2}{2}}, \sqrt{\frac{a_1^2 + a_2^2}{2}}, a_3, \ldots, a_n, \sqrt{\frac{b_1^2 + b_2^2}{2}}, \sqrt{\frac{b_1^2 + b_2^2}{2}}, b_3, \ldots, b_n\right) \quad \blacksquare$$

THEOREM (Cauchy - Schwartz): For any $a_1, \ldots, a_n \geq 0$ and any $b_1, \ldots, b_n \geq 0$ we have

$$(a_1^2 + \ldots + a_n^2)(b_1^2 + \ldots + b_n^2) \geq (a_1 b_1 + \ldots + a_n b_n)^2$$

PROOF: Put

$$f(x_1, \ldots, x_n, y_1, \ldots, y_n) = (x_1^2 + \ldots + x_n^2)(y_1^2 + \ldots + y_n^2) - (x_1 y_1 + \ldots + x_n y_n)^2$$

Assume to the contrary that this function is < 0 for some $x_1, \ldots, x_n, y_1, \ldots, y_n \in [A, B]$. Suppose that f attains its minimal value on $[A, B]$ at some $a_1, \ldots, a_n, b_1, \ldots, b_n \in [A, B]$ with $a_1 \neq a_2$ or $b_1 \neq b_2$. This gives us a contradiction, since

$$f(a_1, a_2, a_3, \ldots, a_n, b_1, b_2, b_3, \ldots, b_n)$$

$$> f\left(\sqrt{\frac{a_1^2 + a_2^2}{2}}, \sqrt{\frac{a_1^2 + a_2^2}{2}}, a_3, \ldots, a_n, \sqrt{\frac{b_1^2 + b_2^2}{2}}, \sqrt{\frac{b_1^2 + b_2^2}{2}}, b_3, \ldots, b_n\right) \quad \blacksquare$$

THEOREM (Cauchy - Schwartz): For any $a_1, \ldots, a_n \geq 0$ and any $b_1, \ldots, b_n \geq 0$ we have

$$(a_1^2 + \ldots + a_n^2)(b_1^2 + \ldots + b_n^2) \geq (a_1 b_1 + \ldots + a_n b_n)^2$$

PROOF: Put

$$\begin{aligned} f(x_1, \ldots, x_n, y_1, \ldots, y_n) = \; & (x_1^2 + \ldots + x_n^2)(y_1^2 + \ldots + y_n^2) \\ & -(x_1 y_1 + \ldots + x_n y_n)^2 \end{aligned}$$

Assume to the contrary that this function is < 0 for some $x_1, \ldots, x_n, y_1, \ldots, y_n \in [A, B]$.

Suppose that f attains its minimal value on $[A, B]$ at some $a_1, \ldots, a_n, b_1, \ldots, b_n \in [A, B]$ with $a_1 \neq a_2$ or $b_1 \neq b_2$. This gives us a contradiction, since

$$f(a_1, a_2, a_3, \ldots, a_n, b_1, b_2, b_3, \ldots, b_n)$$

$$> f\left(\sqrt{\frac{a_1^2 + a_2^2}{2}}, \sqrt{\frac{a_1^2 + a_2^2}{2}}, a_3, \ldots, a_n, \sqrt{\frac{b_1^2 + b_2^2}{2}}, \sqrt{\frac{b_1^2 + b_2^2}{2}}, b_3, \ldots, b_n\right) \quad \blacksquare$$

$$f(a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n)$$

$$= (a_1^2 + a_2^2 + \ldots + a_n^2)(b_1^2 + b_2^2 + \ldots + b_n^2) - (a_1 b_1 + a_2 b_2 + \ldots + a_n b_n)^2$$

THEOREM (Cauchy - Schwartz): For any $a_1, \ldots, a_n \geq 0$ and any $b_1, \ldots, b_n \geq 0$ we have

$$(a_1^2 + \ldots + a_n^2)(b_1^2 + \ldots + b_n^2) \geq (a_1 b_1 + \ldots + a_n b_n)^2$$

PROOF: Put

$$f(x_1, \ldots, x_n, y_1, \ldots, y_n) = (x_1^2 + \ldots + x_n^2)(y_1^2 + \ldots + y_n^2) - (x_1 y_1 + \ldots + x_n y_n)^2$$

Assume to the contrary that this function is < 0 for some $x_1, \ldots, x_n, y_1, \ldots, y_n \in [A, B]$.

Suppose that f attains its minimal value on $[A, B]$ at some $a_1, \ldots, a_n, b_1, \ldots, b_n \in [A, B]$ with $a_1 \neq a_2$ or $b_1 \neq b_2$. This gives us a contradiction, since

$$f(a_1, a_2, a_3, \ldots, a_n, b_1, b_2, b_3, \ldots, b_n)$$

$$> f\left(\sqrt{\frac{a_1^2 + a_2^2}{2}}, \sqrt{\frac{a_1^2 + a_2^2}{2}}, a_3, \ldots, a_n, \sqrt{\frac{b_1^2 + b_2^2}{2}}, \sqrt{\frac{b_1^2 + b_2^2}{2}}, b_3, \ldots, b_n\right) \quad \blacksquare$$

$$f(a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n)$$

$$= (a_1^2 + a_2^2 + \ldots + a_n^2)(b_1^2 + b_2^2 + \ldots + b_n^2) - (a_1 b_1 + a_2 b_2 + \ldots + a_n b_n)^2$$

$$f\left(\sqrt{\frac{a_1^2 + a_2^2}{2}}, \sqrt{\frac{a_1^2 + a_2^2}{2}}, a_3, \ldots, a_n, \sqrt{\frac{b_1^2 + b_2^2}{2}}, \sqrt{\frac{b_1^2 + b_2^2}{2}}, b_3, \ldots, b_n\right)$$

THEOREM (Cauchy - Schwartz): For any $a_1, \ldots, a_n \geq 0$ and any $b_1, \ldots, b_n \geq 0$ we have

$$(a_1^2 + \ldots + a_n^2)(b_1^2 + \ldots + b_n^2) \geq (a_1 b_1 + \ldots + a_n b_n)^2$$

PROOF: Put

$$f(x_1, \ldots, x_n, y_1, \ldots, y_n) = (x_1^2 + \ldots + x_n^2)(y_1^2 + \ldots + y_n^2) - (x_1 y_1 + \ldots + x_n y_n)^2$$

Assume to the contrary that this function is < 0 for some $x_1, \ldots, x_n, y_1, \ldots, y_n \in [A, B]$.

Suppose that f attains its minimal value on $[A, B]$ at some $a_1, \ldots, a_n, b_1, \ldots, b_n \in [A, B]$ with $a_1 \neq a_2$ or $b_1 \neq b_2$. This gives us a contradiction, since

$$f(a_1, a_2, a_3, \ldots, a_n, b_1, b_2, b_3, \ldots, b_n)$$

$$> f\left(\sqrt{\frac{a_1^2 + a_2^2}{2}}, \sqrt{\frac{a_1^2 + a_2^2}{2}}, a_3, \ldots, a_n, \sqrt{\frac{b_1^2 + b_2^2}{2}}, \sqrt{\frac{b_1^2 + b_2^2}{2}}, b_3, \ldots, b_n\right) \quad \blacksquare$$

$$f(a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n)$$

$$= (a_1^2 + a_2^2 + \ldots + a_n^2)(b_1^2 + b_2^2 + \ldots + b_n^2) - (a_1 b_1 + a_2 b_2 + \ldots + a_n b_n)^2$$

$$f\left(\sqrt{\frac{a_1^2 + a_2^2}{2}}, \sqrt{\frac{a_1^2 + a_2^2}{2}}, a_3, \ldots, a_n, \sqrt{\frac{b_1^2 + b_2^2}{2}}, \sqrt{\frac{b_1^2 + b_2^2}{2}}, b_3, \ldots, b_n\right)$$

$$= (a_1^2 + a_2^2 + \ldots + a_n^2)(b_1^2 + b_2^2 + \ldots + b_n^2) -$$

THEOREM (Cauchy - Schwartz): For any $a_1, \ldots, a_n \geq 0$ and any $b_1, \ldots, b_n \geq 0$ we have

$$(a_1^2 + \ldots + a_n^2)(b_1^2 + \ldots + b_n^2) \geq (a_1 b_1 + \ldots + a_n b_n)^2$$

PROOF: Put

$$f(x_1, \ldots, x_n, y_1, \ldots, y_n) = (x_1^2 + \ldots + x_n^2)(y_1^2 + \ldots + y_n^2) - (x_1 y_1 + \ldots + x_n y_n)^2$$

Assume to the contrary that this function is < 0 for some $x_1, \ldots, x_n, y_1, \ldots, y_n \in [A, B]$.

Suppose that f attains its minimal value on $[A, B]$ at some $a_1, \ldots, a_n, b_1, \ldots, b_n \in [A, B]$ with $a_1 \neq a_2$ or $b_1 \neq b_2$. This gives us a contradiction, since

$$f(a_1, a_2, a_3, \ldots, a_n, b_1, b_2, b_3, \ldots, b_n)$$

$$> f\left(\sqrt{\frac{a_1^2 + a_2^2}{2}}, \sqrt{\frac{a_1^2 + a_2^2}{2}}, a_3, \ldots, a_n, \sqrt{\frac{b_1^2 + b_2^2}{2}}, \sqrt{\frac{b_1^2 + b_2^2}{2}}, b_3, \ldots, b_n\right) \quad \blacksquare$$

$$f(a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n)$$

$$= (a_1^2 + a_2^2 + \ldots + a_n^2)(b_1^2 + b_2^2 + \ldots + b_n^2) - (a_1 b_1 + a_2 b_2 + \ldots + a_n b_n)^2$$

$$f\left(\sqrt{\frac{a_1^2 + a_2^2}{2}}, \sqrt{\frac{a_1^2 + a_2^2}{2}}, a_3, \ldots, a_n, \sqrt{\frac{b_1^2 + b_2^2}{2}}, \sqrt{\frac{b_1^2 + b_2^2}{2}}, b_3, \ldots, b_n\right)$$

$$= (a_1^2 + a_2^2 + \ldots + a_n^2)(b_1^2 + b_2^2 + \ldots + b_n^2) - \left(\sqrt{\frac{a_1^2 + a_2^2}{2}}\sqrt{\frac{b_1^2 + b_2^2}{2}} + \sqrt{\frac{a_1^2 + a_2^2}{2}}\sqrt{\frac{b_1^2 + b_2^2}{2}} + \ldots + a_n b_n\right)^2$$

with $a_1 \neq a_2$ or $b_1 \neq b_2$. This gives us a contradiction, since

$$f(a_1, a_2, a_3, \ldots, a_n, b_1, b_2, b_3, \ldots, b_n)$$

$$> f\left(\sqrt{\frac{a_1^2 + a_2^2}{2}}, \sqrt{\frac{a_1^2 + a_2^2}{2}}, a_3, \ldots, a_n, \sqrt{\frac{b_1^2 + b_2^2}{2}}, \sqrt{\frac{b_1^2 + b_2^2}{2}}, b_3, \ldots, b_n\right) \quad \blacksquare$$

$$f(a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n)$$

$$= (a_1^2 + a_2^2 + \ldots + a_n^2)(b_1^2 + b_2^2 + \ldots + b_n^2) - (a_1 b_1 + a_2 b_2 + \ldots + a_n b_n)^2$$

$$f\left(\sqrt{\frac{a_1^2 + a_2^2}{2}}, \sqrt{\frac{a_1^2 + a_2^2}{2}}, a_3, \ldots, a_n, \sqrt{\frac{b_1^2 + b_2^2}{2}}, \sqrt{\frac{b_1^2 + b_2^2}{2}}, b_3, \ldots, b_n\right)$$

$$= (a_1^2 + a_2^2 + \ldots + a_n^2)(b_1^2 + b_2^2 + \ldots + b_n^2) - \left(\sqrt{\frac{a_1^2 + a_2^2}{2}}\sqrt{\frac{b_1^2 + b_2^2}{2}} + \sqrt{\frac{a_1^2 + a_2^2}{2}}\sqrt{\frac{b_1^2 + b_2^2}{2}} + \ldots + a_n b_n\right)^2$$

$$a_1b_1 + a_2b_2 < 2\sqrt{\frac{a_1^2+a_2^2}{2}}\sqrt{\frac{b_1^2+b_2^2}{2}}$$

with $a_1 \neq a_2$ or $b_1 \neq b_2$. This gives us a contradiction, since

$$f(a_1, a_2, a_3, \ldots, a_n, b_1, b_2, b_3, \ldots, b_n)$$

$$> f\left(\sqrt{\frac{a_1^2+a_2^2}{2}}, \sqrt{\frac{a_1^2+a_2^2}{2}}, a_3, \ldots, a_n, \sqrt{\frac{b_1^2+b_2^2}{2}}, \sqrt{\frac{b_1^2+b_2^2}{2}}, b_3, \ldots, b_n\right) \quad \blacksquare$$

$$f(a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n)$$

$$= (a_1^2 + a_2^2 + \ldots + a_n^2)(b_1^2 + b_2^2 + \ldots + b_n^2) - (a_1b_1 + a_2b_2 + \ldots + a_nb_n)^2$$

$$f\left(\sqrt{\frac{a_1^2+a_2^2}{2}}, \sqrt{\frac{a_1^2+a_2^2}{2}}, a_3, \ldots, a_n, \sqrt{\frac{b_1^2+b_2^2}{2}}, \sqrt{\frac{b_1^2+b_2^2}{2}}, b_3, \ldots, b_n\right)$$

$$= (a_1^2 + a_2^2 + \ldots + a_n^2)(b_1^2 + b_2^2 + \ldots + b_n^2) - \left(\sqrt{\frac{a_1^2+a_2^2}{2}}\sqrt{\frac{b_1^2+b_2^2}{2}} + \sqrt{\frac{a_1^2+a_2^2}{2}}\sqrt{\frac{b_1^2+b_2^2}{2}} + \ldots + a_nb_n\right)^2$$

$$a_1b_1 + a_2b_2 < 2\sqrt{\frac{a_1^2+a_2^2}{2}}\sqrt{\frac{b_1^2+b_2^2}{2}}$$

$$(a_1b_1 + a_2b_2)^2 < 4\,\frac{a_1^2+a_2^2}{2}\,\frac{b_1^2+b_2^2}{2}$$

with $a_1 \neq a_2$ or $b_1 \neq b_2$. This gives us a contradiction, since

$$f(a_1, a_2, a_3, \ldots, a_n, b_1, b_2, b_3, \ldots, b_n)$$

$$> f\left(\sqrt{\frac{a_1^2+a_2^2}{2}}, \sqrt{\frac{a_1^2+a_2^2}{2}}, a_3, \ldots, a_n, \sqrt{\frac{b_1^2+b_2^2}{2}}, \sqrt{\frac{b_1^2+b_2^2}{2}}, b_3, \ldots, b_n\right) \quad \blacksquare$$

$$f(a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n)$$

$$= (a_1^2 + a_2^2 + \ldots + a_n^2)(b_1^2 + b_2^2 + \ldots + b_n^2) - (a_1b_1 + a_2b_2 + \ldots + a_nb_n)^2$$

$$f\left(\sqrt{\frac{a_1^2+a_2^2}{2}}, \sqrt{\frac{a_1^2+a_2^2}{2}}, a_3, \ldots, a_n, \sqrt{\frac{b_1^2+b_2^2}{2}}, \sqrt{\frac{b_1^2+b_2^2}{2}}, b_3, \ldots, b_n\right)$$

$$= (a_1^2 + a_2^2 + \ldots + a_n^2)(b_1^2 + b_2^2 + \ldots + b_n^2) - \left(\sqrt{\frac{a_1^2+a_2^2}{2}}\sqrt{\frac{b_1^2+b_2^2}{2}} + \sqrt{\frac{a_1^2+a_2^2}{2}}\sqrt{\frac{b_1^2+b_2^2}{2}} + \ldots + a_nb_n\right)^2$$

$$a_1b_1 + a_2b_2 < 2\sqrt{\frac{a_1^2 + a_2^2}{2}}\sqrt{\frac{b_1^2 + b_2^2}{2}}$$

$$(a_1b_1 + a_2b_2)^2 < 4\,\frac{a_1^2 + a_2^2}{2}\,\frac{b_1^2 + b_2^2}{2}$$

$$(a_1b_1 + a_2b_2)^2 < (a_1^2 + a_2^2)(b_1^2 + b_2^2)$$

with $a_1 \neq a_2$ or $b_1 \neq b_2$. This gives us a contradiction, since

$$f(a_1, a_2, a_3, \ldots, a_n, b_1, b_2, b_3, \ldots, b_n)$$

$$> f\left(\sqrt{\frac{a_1^2 + a_2^2}{2}}, \sqrt{\frac{a_1^2 + a_2^2}{2}}, a_3, \ldots, a_n, \sqrt{\frac{b_1^2 + b_2^2}{2}}, \sqrt{\frac{b_1^2 + b_2^2}{2}}, b_3, \ldots, b_n\right) \quad \blacksquare$$

$$f(a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n)$$

$$= (a_1^2 + a_2^2 + \ldots + a_n^2)(b_1^2 + b_2^2 + \ldots + b_n^2) - (a_1b_1 + a_2b_2 + \ldots + a_nb_n)^2$$

$$f\left(\sqrt{\frac{a_1^2 + a_2^2}{2}}, \sqrt{\frac{a_1^2 + a_2^2}{2}}, a_3, \ldots, a_n, \sqrt{\frac{b_1^2 + b_2^2}{2}}, \sqrt{\frac{b_1^2 + b_2^2}{2}}, b_3, \ldots, b_n\right)$$

$$= (a_1^2 + a_2^2 + \ldots + a_n^2)(b_1^2 + b_2^2 + \ldots + b_n^2) - \left(\sqrt{\frac{a_1^2 + a_2^2}{2}}\sqrt{\frac{b_1^2 + b_2^2}{2}} + \sqrt{\frac{a_1^2 + a_2^2}{2}}\sqrt{\frac{b_1^2 + b_2^2}{2}} + \ldots + a_nb_n\right)^2$$

$$a_1b_1 + a_2b_2 < 2\sqrt{\frac{a_1^2+a_2^2}{2}}\sqrt{\frac{b_1^2+b_2^2}{2}}$$

$$(a_1b_1 + a_2b_2)^2 < 4\frac{a_1^2+a_2^2}{2}\frac{b_1^2+b_2^2}{2}$$

$$(a_1b_1 + a_2b_2)^2 < (a_1^2+a_2^2)(b_1^2+b_2^2)$$

$$a_1^2b_1^2 + 2a_1b_1a_2b_2 + a_2^2b_2^2 < a_1^2b_1^2 + a_1^2b_2^2 + a_2^2b_1^2 + a_2^2b_2^2$$

with $a_1 \neq a_2$ or $b_1 \neq b_2$. This gives us a contradiction, since

$$f(a_1, a_2, a_3, \ldots, a_n, b_1, b_2, b_3, \ldots, b_n)$$
$$> f\left(\sqrt{\frac{a_1^2+a_2^2}{2}}, \sqrt{\frac{a_1^2+a_2^2}{2}}, a_3, \ldots, a_n, \sqrt{\frac{b_1^2+b_2^2}{2}}, \sqrt{\frac{b_1^2+b_2^2}{2}}, b_3, \ldots, b_n\right) \quad \blacksquare$$

$$f(a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n)$$
$$= (a_1^2 + a_2^2 + \ldots + a_n^2)(b_1^2 + b_2^2 + \ldots + b_n^2) - (a_1b_1 + a_2b_2 + \ldots + a_nb_n)^2$$

$$f\left(\sqrt{\frac{a_1^2+a_2^2}{2}}, \sqrt{\frac{a_1^2+a_2^2}{2}}, a_3, \ldots, a_n, \sqrt{\frac{b_1^2+b_2^2}{2}}, \sqrt{\frac{b_1^2+b_2^2}{2}}, b_3, \ldots, b_n\right)$$
$$= (a_1^2 + a_2^2 + \ldots + a_n^2)(b_1^2 + b_2^2 + \ldots + b_n^2) - \left(\sqrt{\frac{a_1^2+a_2^2}{2}}\sqrt{\frac{b_1^2+b_2^2}{2}} + \sqrt{\frac{a_1^2+a_2^2}{2}}\sqrt{\frac{b_1^2+b_2^2}{2}} + \ldots + a_nb_n\right)^2$$

$$a_1b_1 + a_2b_2 < 2\sqrt{\frac{a_1^2 + a_2^2}{2}}\sqrt{\frac{b_1^2 + b_2^2}{2}} \qquad 2a_1b_1a_2b_2 < a_1^2b_2^2 + a_2^2b_1^2$$

$$(a_1b_1 + a_2b_2)^2 < 4\frac{a_1^2 + a_2^2}{2}\frac{b_1^2 + b_2^2}{2}$$

$$(a_1b_1 + a_2b_2)^2 < (a_1^2 + a_2^2)(b_1^2 + b_2^2)$$

$$a_1^2b_1^2 + 2a_1b_1a_2b_2 + a_2^2b_2^2 < a_1^2b_1^2 + a_1^2b_2^2 + a_2^2b_1^2 + a_2^2b_2^2$$

with $a_1 \neq a_2$ or $b_1 \neq b_2$. This gives us a contradiction, since

$$f(a_1, a_2, a_3, \ldots, a_n, b_1, b_2, b_3, \ldots, b_n)$$

$$> f\left(\sqrt{\frac{a_1^2 + a_2^2}{2}}, \sqrt{\frac{a_1^2 + a_2^2}{2}}, a_3, \ldots, a_n, \sqrt{\frac{b_1^2 + b_2^2}{2}}, \sqrt{\frac{b_1^2 + b_2^2}{2}}, b_3, \ldots, b_n\right) \quad \blacksquare$$

$$f(a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n)$$

$$= (a_1^2 + a_2^2 + \ldots + a_n^2)(b_1^2 + b_2^2 + \ldots + b_n^2) - (a_1b_1 + a_2b_2 + \ldots + a_nb_n)^2$$

$$f\left(\sqrt{\frac{a_1^2 + a_2^2}{2}}, \sqrt{\frac{a_1^2 + a_2^2}{2}}, a_3, \ldots, a_n, \sqrt{\frac{b_1^2 + b_2^2}{2}}, \sqrt{\frac{b_1^2 + b_2^2}{2}}, b_3, \ldots, b_n\right)$$

$$= (a_1^2 + a_2^2 + \ldots + a_n^2)(b_1^2 + b_2^2 + \ldots + b_n^2) - \left(\sqrt{\frac{a_1^2 + a_2^2}{2}}\sqrt{\frac{b_1^2 + b_2^2}{2}} + \sqrt{\frac{a_1^2 + a_2^2}{2}}\sqrt{\frac{b_1^2 + b_2^2}{2}} + \ldots + a_nb_n\right)^2$$

$$a_1b_1 + a_2b_2 < 2\sqrt{\frac{a_1^2 + a_2^2}{2}}\sqrt{\frac{b_1^2 + b_2^2}{2}}$$

$$(a_1b_1 + a_2b_2)^2 < 4\,\frac{a_1^2 + a_2^2}{2}\,\frac{b_1^2 + b_2^2}{2}$$

$$(a_1b_1 + a_2b_2)^2 < (a_1^2 + a_2^2)(b_1^2 + b_2^2)$$

$$a_1^2b_1^2 + 2a_1b_1a_2b_2 + a_2^2b_2^2 < a_1^2b_1^2 + a_1^2b_2^2 + a_2^2b_1^2 + a_2^2b_2^2$$

$$2a_1b_1a_2b_2 < a_1^2b_2^2 + a_2^2b_1^2$$

$$0 < a_1^2b_2^2 - 2a_1b_1a_2b_2 + a_2^2b_1^2$$

with $a_1 \neq a_2$ or $b_1 \neq b_2$. This gives us a contradiction, since

$$f(a_1, a_2, a_3, \ldots, a_n, b_1, b_2, b_3, \ldots, b_n)$$

$$> f\left(\sqrt{\frac{a_1^2 + a_2^2}{2}}, \sqrt{\frac{a_1^2 + a_2^2}{2}}, a_3, \ldots, a_n, \sqrt{\frac{b_1^2 + b_2^2}{2}}, \sqrt{\frac{b_1^2 + b_2^2}{2}}, b_3, \ldots, b_n\right) \quad \blacksquare$$

$$f(a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n)$$

$$= (a_1^2 + a_2^2 + \ldots + a_n^2)(b_1^2 + b_2^2 + \ldots + b_n^2) - (a_1b_1 + a_2b_2 + \ldots + a_nb_n)^2$$

$$f\left(\sqrt{\frac{a_1^2 + a_2^2}{2}}, \sqrt{\frac{a_1^2 + a_2^2}{2}}, a_3, \ldots, a_n, \sqrt{\frac{b_1^2 + b_2^2}{2}}, \sqrt{\frac{b_1^2 + b_2^2}{2}}, b_3, \ldots, b_n\right)$$

$$= (a_1^2 + a_2^2 + \ldots + a_n^2)(b_1^2 + b_2^2 + \ldots + b_n^2) - \left(\sqrt{\frac{a_1^2 + a_2^2}{2}}\sqrt{\frac{b_1^2 + b_2^2}{2}} + \sqrt{\frac{a_1^2 + a_2^2}{2}}\sqrt{\frac{b_1^2 + b_2^2}{2}} + \ldots + a_nb_n\right)^2$$

$$a_1b_1 + a_2b_2 < 2\sqrt{\frac{a_1^2+a_2^2}{2}}\sqrt{\frac{b_1^2+b_2^2}{2}}$$

$$(a_1b_1 + a_2b_2)^2 < 4\,\frac{a_1^2+a_2^2}{2}\,\frac{b_1^2+b_2^2}{2}$$

$$(a_1b_1 + a_2b_2)^2 < (a_1^2+a_2^2)(b_1^2+b_2^2)$$

$$a_1^2b_1^2 + 2a_1b_1a_2b_2 + a_2^2b_2^2 < a_1^2b_1^2 + a_1^2b_2^2 + a_2^2b_1^2 + a_2^2b_2^2$$

$$2a_1b_1a_2b_2 < a_1^2b_2^2 + a_2^2b_1^2$$

$$0 < a_1^2b_2^2 - 2a_1b_1a_2b_2 + a_2^2b_1^2$$

$$0 < (a_1b_2 - a_2b_1)^2$$

with $a_1 \neq a_2$ or $b_1 \neq b_2$. This gives us a contradiction, since

$$f(a_1, a_2, a_3, \ldots, a_n, b_1, b_2, b_3, \ldots, b_n)$$

$$> f\left(\sqrt{\frac{a_1^2+a_2^2}{2}}, \sqrt{\frac{a_1^2+a_2^2}{2}}, a_3, \ldots, a_n, \sqrt{\frac{b_1^2+b_2^2}{2}}, \sqrt{\frac{b_1^2+b_2^2}{2}}, b_3, \ldots, b_n\right) \quad \blacksquare$$

$$f(a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n)$$

$$= (a_1^2 + a_2^2 + \ldots + a_n^2)(b_1^2 + b_2^2 + \ldots + b_n^2) - (a_1b_1 + a_2b_2 + \ldots + a_nb_n)^2$$

$$f\left(\sqrt{\frac{a_1^2+a_2^2}{2}}, \sqrt{\frac{a_1^2+a_2^2}{2}}, a_3, \ldots, a_n, \sqrt{\frac{b_1^2+b_2^2}{2}}, \sqrt{\frac{b_1^2+b_2^2}{2}}, b_3, \ldots, b_n\right)$$

$$= (a_1^2 + a_2^2 + \ldots + a_n^2)(b_1^2 + b_2^2 + \ldots + b_n^2) - \left(\sqrt{\frac{a_1^2+a_2^2}{2}}\sqrt{\frac{b_1^2+b_2^2}{2}} + \sqrt{\frac{a_1^2+a_2^2}{2}}\sqrt{\frac{b_1^2+b_2^2}{2}} + \ldots + a_nb_n\right)^2$$

$$a_1b_1 + a_2b_2 < 2\sqrt{\frac{a_1^2 + a_2^2}{2}}\sqrt{\frac{b_1^2 + b_2^2}{2}}$$

$$(a_1b_1 + a_2b_2)^2 < 4\,\frac{a_1^2 + a_2^2}{2}\,\frac{b_1^2 + b_2^2}{2}$$

$$(a_1b_1 + a_2b_2)^2 < (a_1^2 + a_2^2)(b_1^2 + b_2^2)$$

$$a_1^2b_1^2 + 2a_1b_1a_2b_2 + a_2^2b_2^2 < a_1^2b_1^2 + a_1^2b_2^2 + a_2^2b_1^2 + a_2^2b_2^2$$

$$2a_1b_1a_2b_2 < a_1^2b_2^2 + a_2^2b_1^2$$

$$0 < a_1^2b_2^2 - 2a_1b_1a_2b_2 + a_2^2b_1^2$$

$$0 < (a_1b_2 - a_2b_1)^2$$

with $a_1b_2 \neq a_2b_1$. This gives us a contradiction, since

$$f(a_1, a_2, a_3, \ldots, a_n, b_1, b_2, b_3, \ldots, b_n)$$
$$> f\left(\sqrt{\frac{a_1^2 + a_2^2}{2}}, \sqrt{\frac{a_1^2 + a_2^2}{2}}, a_3, \ldots, a_n, \sqrt{\frac{b_1^2 + b_2^2}{2}}, \sqrt{\frac{b_1^2 + b_2^2}{2}}, b_3, \ldots, b_n\right) \quad \blacksquare$$

$$f(a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n)$$
$$= (a_1^2 + a_2^2 + \ldots + a_n^2)(b_1^2 + b_2^2 + \ldots + b_n^2) - (a_1b_1 + a_2b_2 + \ldots + a_nb_n)^2$$

$$f\left(\sqrt{\frac{a_1^2 + a_2^2}{2}}, \sqrt{\frac{a_1^2 + a_2^2}{2}}, a_3, \ldots, a_n, \sqrt{\frac{b_1^2 + b_2^2}{2}}, \sqrt{\frac{b_1^2 + b_2^2}{2}}, b_3, \ldots, b_n\right)$$
$$= (a_1^2 + a_2^2 + \ldots + a_n^2)(b_1^2 + b_2^2 + \ldots + b_n^2) - \left(\sqrt{\frac{a_1^2 + a_2^2}{2}}\sqrt{\frac{b_1^2 + b_2^2}{2}} + \sqrt{\frac{a_1^2 + a_2^2}{2}}\sqrt{\frac{b_1^2 + b_2^2}{2}} + \ldots + a_nb_n\right)^2$$

THEOREM (Cauchy - Schwartz): For any $a_1, \ldots, a_n \geq 0$ and any $b_1, \ldots, b_n \geq 0$ we have

$$(a_1^2 + \ldots + a_n^2)(b_1^2 + \ldots + b_n^2) \geq (a_1 b_1 + \ldots + a_n b_n)^2$$

PROOF: Put

$$f(x_1, \ldots, x_n, y_1, \ldots, y_n) = (x_1^2 + \ldots + x_n^2)(y_1^2 + \ldots + y_n^2) - (x_1 y_1 + \ldots + x_n y_n)^2$$

Assume to the contrary that this function is <0 for some $x_1, \ldots, x_n, y_1, \ldots, y_n \in [A, B]$.

Suppose that f attains its minimal value on $[A, B]$ at some $a_1, \ldots, a_n, b_1, \ldots, b_n \in [A, B]$ with $a_1 b_2 \neq a_2 b_1$. This gives us a contradiction, since

$$f(a_1, a_2, a_3, \ldots, a_n, b_1, b_2, b_3, \ldots, b_n)$$

$$> f\left(\sqrt{\frac{a_1^2 + a_2^2}{2}}, \sqrt{\frac{a_1^2 + a_2^2}{2}}, a_3, \ldots, a_n, \sqrt{\frac{b_1^2 + b_2^2}{2}}, \sqrt{\frac{b_1^2 + b_2^2}{2}}, b_3, \ldots, b_n\right) \quad \blacksquare$$

$$f(a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n)$$

$$= (a_1^2 + a_2^2 + \ldots + a_n^2)(b_1^2 + b_2^2 + \ldots + b_n^2) - (a_1 b_1 + a_2 b_2 + \ldots + a_n b_n)^2$$

$$f\left(\sqrt{\frac{a_1^2 + a_2^2}{2}}, \sqrt{\frac{a_1^2 + a_2^2}{2}}, a_3, \ldots, a_n, \sqrt{\frac{b_1^2 + b_2^2}{2}}, \sqrt{\frac{b_1^2 + b_2^2}{2}}, b_3, \ldots, b_n\right)$$

$$= (a_1^2 + a_2^2 + \ldots + a_n^2)(b_1^2 + b_2^2 + \ldots + b_n^2) - \left(\sqrt{\frac{a_1^2 + a_2^2}{2}}\sqrt{\frac{b_1^2 + b_2^2}{2}} + \sqrt{\frac{a_1^2 + a_2^2}{2}}\sqrt{\frac{b_1^2 + b_2^2}{2}} + \ldots + a_n b_n\right)^2$$

THEOREM (Cauchy - Schwartz): For any $a_1, \ldots, a_n \geq 0$ and any $b_1, \ldots, b_n \geq 0$ we have

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PROOF: Put

$$f(x_1, \ldots, x_n, y_1, \ldots, y_n) = (x_1^p + \ldots + x_n^p)^{1/p}(y_1^q + \ldots + y_n^q)^{1/q} - (x_1 y_1 + \ldots + x_n y_n)$$

Assume to the contrary that this function is < 0 for some $x_1, \ldots, x_n, y_1, \ldots, y_n \in [A, B]$. Suppose that f attains its minimal value on $[A, B]$ at some $a_1, \ldots, a_n, b_1, \ldots, b_n \in [A, B]$ with $a_1 b_2 \neq a_2 b_1$. This gives us a contradiction, since

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$$a_1 b_1 + a_2 b_2 < 2\sqrt[p]{\frac{a_1^p + a_2^p}{2}}\sqrt[q]{\frac{b_1^q + b_2^q}{2}}$$

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THEOREM: For any $a, b, c > 0$ we have

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CONJECTURE (Shapiro, 1954): For any $a_1, a_2, \ldots, a_n > 0$ **we have**

$$\frac{a_1}{a_2+a_3} + \frac{a_2}{a_3+a_4} + \ldots + \frac{a_{n-1}}{a_n+a_1} + \frac{a_n}{a_1+a_2} \geq \frac{n}{2}$$

THEOREM: For any $a, b, c > 0$ **we have**

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}$$

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THEOREM (Mitrinovic, 1993): For any $a_1, a_2, \ldots, a_n > 0$ **we have**

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if $n = 3$

THEOREM: For any $a, b, c > 0$ we have

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}$$

THEOREM: For any $a, b, c, d > 0$ we have

$$\frac{a}{b+c} + \frac{b}{c+d} + \frac{c}{d+a} + \frac{d}{a+b} \geq 2$$

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if $n = 3, 4$

THEOREM: For any $a, b, c > 0$ we have

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}$$

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if $n = 3, 4, 5$

THEOREM: For any $a, b, c > 0$ we have

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}$$

THEOREM: For any $a, b, c, d > 0$ we have

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if $n = 3, 4, 5, 6$

THEOREM: For any $a, b, c > 0$ we have

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if $n = 3, 4, 5, 6, 7$

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if $n = 3, 4, 5, 6, 7, 8$

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if $n = 3, 4, 5, 6, 7, 8, 9$

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if $n = 3, 4, 5, 6, 7, 8, 9, 10$

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if $n = 3, 4, 5, 6, 7, 8, 9, 10, 11$

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if $n = 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$

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if $n = 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 15$

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if $n = 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 15, 17, 19, 21, 23.$